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String Cosmologies

SZCZECIN 2002

Contents

Preface

Rapid development of the theory of elementary particles together with experimental limitations of biggest world accelerators made particle physicist grow their interest in astrophysics and cosmology. This is mainly due to the discovery of the microwave background radiation in 1965 which confirmed that the universe was hot and dense in the past and so it could serve itself as a big accelerator of particles. These particles possessed huge energies which were naturally redshifted during the evolution, but their remnants can be observed in the present-day universe by telescopes or other types of astronomical detectors.

First attempt to combine elementary particles and cosmology was given in the seventies and this is what we now call standard model of hot universe. This model, however, did not explain the observed isotropy of the microwave background and had to be appended by the idea of inflation. Inflation as a rapid accelerated expansion of the universe caused by potential energy of a scalar field (inflaton) referred as high as to GUT scale, but it did not include gravity in one scheme. The inclusion of gravity became possible once the idea of unification of gauge interactions and gravity in superstring theory appeared. The idea of getting elementary particles out of modes of a single fundamental object – a fundamental string – has an appealing beauty comparable to a Mendeleev scheme for elements. Although superstring theory is still in his early days, it has proved successful in many aspects. It naturally includes quantum gravity since it contains graviton in its spectrum, for example. It allows to cancel divergencies for gravity – not an easy task within the standard framework in view of a dimensionful gravitational constant.

However, superstring theory lacks one point. Its experimental basis is empty. The only support for the theory is the fact of getting the Einstein limit – not a very big deal.

In the early nineties Gasperini and Veneziano started regular programme of studying the cosmological aspects of superstring theories. They elaborated the whole scenario of the evolution for the universe based on the low-energy-effective actions for superstring theories now called pre-big-bang scenario which we discuss in Chapter 3. There is no doubt that this is an alternative inflationary scenario to inflationary scenarios of the eighties. The main difference is that inflation in pre-big-bang models is driven by kinetic energy of the scalar field (a 'stringy' mode) dilaton rather than by potential energy of the inflaton. Besides, many observational aspects of pre-big-bang (CMBR anisotropies, power spectrum of density perturbations, gravitational wave spectrum etc.) have been elaborated which are now ready to test in cosmology. The main drawback of pre-big-bang is "graceful-exit" from pre-big-bang phase to post-big-bang phase which we address in our Chapter 4.

In order to be precise one should emphasize that superstring or M-theory cosmology is a broader field than just pre-big-bang cosmology since the validity of the truncated effective-actions is limited. Since 1999 a rapidly growing field became brane cosmology (or Randall-Sundrum cosmology) and parallel to it M-theory cosmology (Hořava-Witten, bosonic M-theory). These topics will be covered in Chapter 5.

Finally, I would like to mention some other points which refer to my personal interest in superstring cosmologies. Firstly, I started my scientific career at the Institute of Theoretical Physics of the University of Wrocław where superstring theory was very popular. Quite differently, cosmology was not very much appreciated (if at all), so that at some stage I felt very happy I was able to find a way to connect the two. Secondly, in the mid nineties I visited at University of Sussex, where there formed a small group of people who started working on the topic, and I simply joined. Thirdly, it is not a very well-known fact that the creator of an anticommuting algebra – Hermann Günter Grassmann – used to work and live in my home city Szczecin in the 19th century. Believe me or not, I know people who think that Hermann Günter Grassmann and Marcel Grossmann (of Marcel Grossmann Meetings in relativity) are the same individuals. That is not simply true.

Last but not least, I wish to thank the people who gave me enlightening hints about superstring theory and cosmology. These are: John Barrow, Edmund Copeland, Nemanja Kaloper, Claus Kiefer, Kerstin Kunze, Amithaba Lahiri, Arne Larsen, Jim Lidsey, Krzysztof Meissner, Sergey Roshchupkin, Carlo Ungarelli, Gabriele Veneziano, David Wands and Alexander Zheltukhin. I thank my student Izabela Próchnicka for help in preparing some figures. The research projects which led to the results contained in this book were partially supported by Royal Society, NATO and DAAD grants as well as by Polish Research Committee (KBN) Grants No 2P03B 196 10 and 2PO3B 105 16.

In Szczecin, May 2001

Chapter 1

Introduction

1.1 Superstring theory

String theory is one of the best candidates for unification of gauge interactions and gravity and deals with extended objects instead of point particles [110]. It is to be applied at the energy scale which is at or above the Planck scale $l_{pl} = \sqrt{G\hbar/c^3} = 10^{-33}$cm, where G is the Newton constant, $\hbar$ the Planck constant, and c the velocity of light.

The second quantized action for string theory is not yet known. The first quantization of a string results in a 2-dimensional second quantized point particle quantum field theory on the world–sheet of a string. This is because the string position $X^\mu(\tau, \sigma)$ and its momentum $p^\mu(\tau, \sigma)$ are the functions of string timelike and spacelike coordinates τ and σ and, after quantization, they become operators – these operators can be viewed as quantized fields of an ordinary point particle theory.

String theory contains a number of massless and massive degrees of freedom depending on whether strings are considered open or closed and what boundary conditions are imposed. In the σ-model approach one takes only massless (or very light) modes into account. After imposing conformal invariance, suitable β-functions of the renormalization group are required to vanish and they can be interpreted as field equations of an appropriate effective action [42]. The most important modes which appear in all effective theories are a scalar field, the dilaton ϕ, a symmetric tensor field, the graviton $g_{\mu\nu}$, and an antisymmetric tensor field, the axion $B_{\mu\nu}$. The unique role is played by the dilaton ϕ since its vacuum expectation value determines the strengths of both gauge and gravitational couplings.

String theory relies on certain number of parameters. These are the string tension

$$T \equiv \frac{1}{2\pi\alpha'}, \tag{1.1.1}$$

where α' is the inverse string tension parameter (Regge slope)$/2\pi$ and it defines the characteristic string length (in units $\hbar = c = 1$)

$$\lambda_s \equiv \sqrt{\alpha'}. \tag{1.1.2}$$

In D-dimensional spacetime the string (gauge) coupling constant $g_s \sim \sqrt{\alpha_{gauge}}$ is given by the dilaton as follows

$$g_s \equiv e^{\phi/2} = \left(\frac{l_{pl}}{\lambda_s}\right)^{\frac{D}{2}-1}, \tag{1.1.3}$$

where l_{pl} is the effective Planck length

$$l_{pl} \equiv \lambda_s \exp\left(\frac{\phi}{D-2}\right). \tag{1.1.4}$$

As mentioned, strings can be either open or closed [110]. In an open string theory right-moving modes have to be the same as left-moving modes (because of the reflection at free ends). The simplest string theory is the bosonic theory – it contains only the bosonic modes and has no supersymmetry (i.e., $N = 0$, where N is the number of supersymmetries). Bosonic string theory can be either open or closed string theory. The string theories which involve also fermionic modes are called superstring theories since they exhibit supersymmetry between bosonic and fermionic coordinates. The open superstring theory is called type I theory and this is $N = 1$ supersymmetric theory. As for closed string theory of type II, left-moving modes and right-moving modes are independent and each sector has $N = 1$ supersymmetry (then the total number of supersymmetries is $N = 2$). The theory in which left-moving modes and right-moving modes have opposite chirality is called type IIA theory and the theory in which left-moving modes and right moving modes have the same chirality is called type IIB theory. The most complicated superstring theory is heterotic theory. In this theory left-moving modes have no supersymmetry, while right-moving modes are supersymmetric, so it is $N = 1$ theory. The left-moving sector requires the gauge groups to be either $SO(32)$ or $E_8 \times E_8$ and due to that one has either heterotic $SO(32)$ or heterotic $E_8 \times E_8$ theories. It is important that bosonic string theories can only be consistently formulated in $D = 26$ spacetime dimensions and superstring theories can be formulated in $D = 10$ spacetime dimensions. All these theories will be discussed in more detail in Chapter 3.

1.2 Supergravity and M-theory

The core of superstring theory seems to be its duality symmetries. These are some extra symmetries of the field equations which, roughly, generalize the symmetry of vacuum (or with electric charges and magnetic monopoles) Maxwell equations with respect to a change $\mathbf{E} \to \mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}$, where $\mathbf{E}$ is the electric and $\mathbf{B}$ magnetic field. More precisely, there is a T-duality symmetry which concludes that any physical process in a box of radius R is equivalent to a dual process in a box of radius $1/R$ (scattering amplitudes are equal [178]). This, in turn, means that in string theory, unlike in relativity, the notion of distance is not an invariant. T-duality has interesting consequences onto the large-scale structure of the universe (cosmology) which is the topic of Chapter 3. Another type of duality, which is S-duality, relates weak coupling limit $g_s \ll 1$ of one (super)string theory with strong coupling limit $g_s' \equiv 1/g_s \gg 1$ of another theory. This strongly suggests that these theories are subject to a more general theory, provisionally called M-theory [177]. It appears that M-theory is not to be the string theory, but rather the theory of some more complicated objects such as 2-dimensional membranes. M-theory has got its low-energy limit which is 11-dimensional supergravity – some points of which will be discussed in Chapter 5.

String theory is a very rapidly developing topic. One of its interesting issues

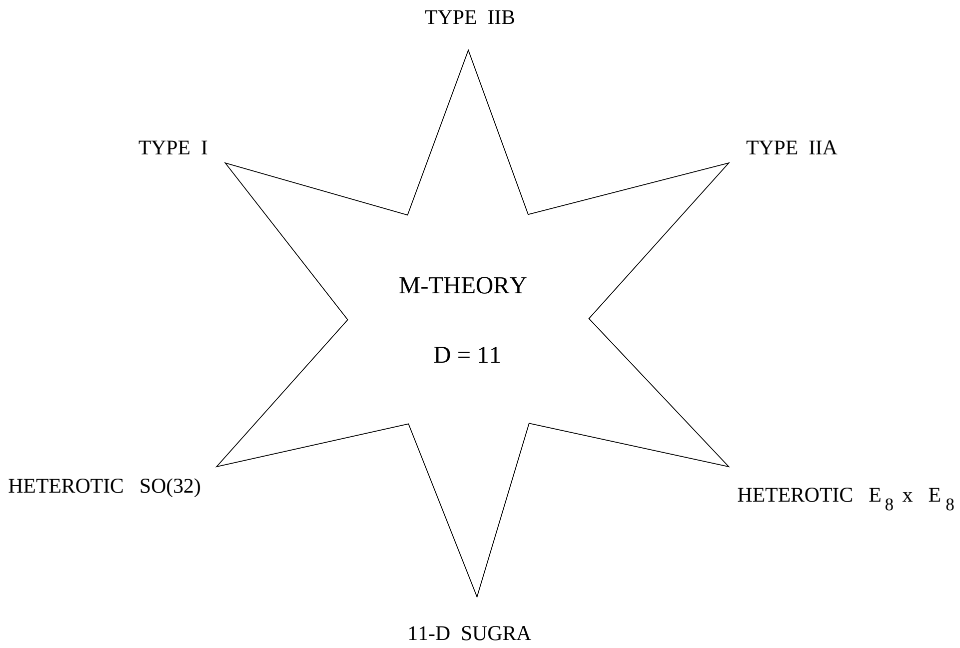

Figure 1.1: Moduli space of superstring theories. All of these theories are connected via different duality symmetries. M-theory is assumed to be their strong coupling limit. Supergravity is a low-energy limit of M-theory

are the Holographic Principle and Maldacena's conjecture. The Holographic Principle says that the degrees of freedom of a certain volume do not reside inside (as it happens in ordinary quantum field theory), but that they rather reside on the boundary of the volume [84, 119]. This takes its motivation from physics of black holes and Hawking radiation. The Maldacena's conjecture [164] says that supergravity/type IIB theory on $AdS_5 \times S_5$ (where AdS_5 is a 5-dimensional anti-de-Sitter space) is equivalent to Super-Yang-Mills theory on the boundary $(S_3 \times R) \times S_5$. String theory is a very advanced mathematical theory. It refers to the energy scale close to the Planck scale 10^{19} Gev and for such scale no experiment is currently available in the accelerators although there is some hope to lower its scale up to the scale of TeV by Randall-Sundrum mechanism [5, 6, 179, 180]. In any case there is a need to elaborate some predictions from string theory onto the large-scale structure of the universe which can be observed as a consequence of an early 'stringy' phase of its evolution. An amazing observational result from type Ia supernovae [176, 183] strongly suggests the existence of an exotic (negative pressure) matter in the universe such as the cosmological constant or other forms of vacuum energy. Superstring theory is one of the theories at hand which can provide us with such type of matter and that is why it is so important to study its cosmological implications. These implications are completely different from those based on general relativity or other alternative theories of gravity. This, especially, refers to the early stages of the evolution of the universe where the expansion was inflationary. It emerges that superstring inspired cosmological models admit a different type of inflation which is kinetic-energy-driven inflation and they lead to the pre-big-bang scenario with different boundary conditions for the universe. These problems will be the main objectives of this book.

Chapter 2

Strings in curved background

2.1 World-sheet action

A free string which propagates in flat Minkowski spacetime sweeps out a world-sheet (2-dimensional surface) in contrast to a point particle, whose history is a world-line. The world-sheet action for a free, closed superstring is given by the formula [178]

$$S = \frac{T}{2}\int d\tau d\sigma \eta_{\mu\nu}\sqrt{-h}\,\Big[h^{ab}\partial_a X^\mu \partial_b X^\nu + i\bar{\psi}^\mu \gamma^a \partial_a \psi^\nu\Big], \tag{2.1.1}$$

where $T = 1/2\pi\alpha'$ is the string tension, τ and σ are the (spacelike and time-like, respectively) string coordinates, h^{ab} is a 2-dimensional world-sheet metric $(a, b = 0, 1)$, $h = det(h_{ab})$, $X^\mu(\tau, \sigma)$ $(\mu, \nu = 0, 1, \dots D - 1)$ are the coordinates of the string world-sheet in D-dimensional Minkowski spacetime with metric $\eta_{\mu\nu}$ and $\partial_a \equiv (\partial/\partial\tau, \partial/\partial\sigma)$. The spinors on the world-sheet are denoted by $\psi^\mu = \psi^\mu(\tau, \sigma)$ and γ^a are 2×2 Dirac matrices satisfying the algebra $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$.

The action (2.1.1) relates the spacetime (bosonic) coordinates $X^\mu(\tau, \sigma)$ to the fermionic coordinates $\psi^\mu = \psi^\mu(\tau, \sigma)$ via the corresponding equations of motion, and it is invariant under infinitesimal supersymmetric transformation

$$\delta X^\mu = i\bar{\epsilon}\psi^\mu, \qquad \delta\psi^\mu = \gamma^a \partial_a X^\mu \epsilon, \tag{2.1.2}$$

where ϵ is a constant anti-commuting spinor.

From now on we restrict ourselves to bosonic coordinates only, but instead of the flat Minkowski background we consider any *curved* spacetime with metric $g_{\mu\nu}$ in which case the action (2.1.1) reduces to

$$S = \frac{T}{2} \int d\tau d\sigma \sqrt{-h} h^{ab} g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu . \qquad (2.1.3)$$

The action (2.1.3) is usually called the Polyakov action [178]. It is fully equivalent the so-called Nambu-Goto action which contains a square root of $h = det(h_{ab})$ and is simply the surface area of the string worldsheet

$$S = T \int d\tau d\sigma \sqrt{-h}. \qquad (2.1.4)$$

It is useful to present the relation between background (target space) metric $g_{\mu\nu}$ and the induced worldsheet metric h_{ab} embedded in $g_{\mu\nu}$ which reads

$$h_{ab} = g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu . \qquad (2.1.5)$$

In (2.1.3) one can then apply the conformal gauge

$$\sqrt{-h} h^{ab} = \eta^{ab} \qquad (2.1.6)$$

which allows the 2-dimensional world-sheet metric h^{ab} to be taken as flat metric η^{ab}. This is because the action is invariant under Weyl (conformal) transformations $h'^{ab} = f(\sigma) h^{ab}$ and the h^{ab}-dependence can be gauged away. However, Weyl transformations rescale invariant intervals, hence there is no invariant notion of distance between two points. In the conformal gauge the action (2.1.3) takes the form

$$S = \frac{T}{2} \int d\tau d\sigma \eta^{ab} g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu . \qquad (2.1.7)$$

In fact, the action (2.1.7) describes a non-trivial quantum field theory (QFT), known as nonlinear σ-model [42, 178].

The variation of the action (2.1.7) gives equations of motion of a tensile string ($T \neq 0$) and the conformal gauge condition (2.1.6) gives the constraints equations.

However, the action (2.1.7) has a disadvantage. Alike the point particle case with its zero mass limit, one cannot take the limit of zero tension $T \to 0$ here. In order to avoid this one has to apply a different action which contains a Lagrange multiplier $E(\tau, \sigma)$ [185, 216]

$$S = \frac{1}{2} \int d\tau d\sigma \left[\frac{g_{\mu\nu} h^{ab} \partial_a X^\mu \partial_b X^\nu}{E^2(\tau, \sigma)} - \frac{E(\tau, \sigma)}{\alpha'^2} \right]. \tag{2.1.8}$$

Varying the action (2.1.8) with respect to E gives the condition

$$E = \alpha' \sqrt{-h}. \tag{2.1.9}$$

Substitution (2.1.9) back into (2.1.8) gives simply the Nambu-Goto action (2.1.4).

By the introduction of a new constant γ with the dimension of $(length)^2$ we define a dimensionless parameter (compare (1.1.2))

$$\varepsilon = \frac{\gamma}{\alpha'}. \tag{2.1.10}$$

Finally, after imposing the gauge

$$E = -\gamma \left(g_{\mu\nu} X'^\mu X'^\nu \right), \tag{2.1.11}$$

together with the orthogonality condition (the velocity of a string must be perpendicular to it)

$$g_{\mu\nu} \dot{X}^\mu X'^\nu = 0, \tag{2.1.12}$$

we get the equations of motion and the constraints for the action (2.1.8) [77, 184, 185, 216]

$$\ddot{X} + \Gamma^\mu_{\nu\rho} \dot{X}^\nu \dot{X}^\rho = \varepsilon^2 (X''^\mu + \Gamma^\mu_{\nu\rho} X'^\nu X'^\rho), \tag{2.1.13}$$

$$g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu = -\varepsilon^2 g_{\mu\nu} X'^\mu X'^\nu, \tag{2.1.14}$$

where $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols of a particular model of spacetime, $(...)^\cdot \equiv \partial/\partial\tau$, $(...)' \equiv \partial/\partial\sigma$, and

$$X^0 = t(\tau, \sigma), \quad X^1 = r(\tau, \sigma), \quad X^2 = \theta(\tau, \sigma), \quad X^3 = \varphi(\tau, \sigma). \tag{2.1.15}$$

Now it makes sense to take the limits:

- $\varepsilon^2 \to 0$ ($T \to 0$) for *tensionless (null)* strings which worldsheet is placed on the light cone,
- $\varepsilon^2 \to 1$ for *tensile* strings which worldsheet is placed inside the light cone
- $\varepsilon = \gamma/\alpha' \ll 1$ for perturbative scheme for the tensile strings expanded out of the null strings [79, 185, 216].

These equations can also be obtained using other types of gauges [33].

An important characteristic for both null and tensile strings is their invariant string size defined by (for closed strings) [178]

$$S(\tau) = \int_0^{2\pi} S(\tau,\sigma)\, d\sigma, \tag{2.1.16}$$

where

$$S(\tau,\sigma) = \sqrt{-g_{\mu\nu} X'^{\mu} X'^{\nu}}\,. \tag{2.1.17}$$

Finally, we present again the set of equations of motion for the classical evolution of strings

$$\ddot{X}^{\mu} + \Gamma^{\mu}_{\nu\rho}\dot{X}^{\nu}\dot{X}^{\rho} = \lambda\left(X''^{\mu} + \Gamma^{\mu}_{\nu\rho}X'^{\nu}X'^{\rho}\right), \tag{2.1.18}$$

and the constraints which read as

$$g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu} = -\lambda g_{\mu\nu}X'^{\mu}X'^{\nu}, \tag{2.1.19}$$

$$g_{\mu\nu}\dot{X}^{\mu}X'^{\nu} = 0, \tag{2.1.20}$$

where we have introduced the parameter $\lambda \equiv \varepsilon^2$, and $\mu, \nu, \rho = 0, 1, 2, 3$ from now on (we will be considered $D = 4$ dimensional spacetimes only). The constraint (2.1.19) is a simple consequence of the application of the conformal gauge (2.1.6), i.e., the conditions $h_{00} + h_{11} = 0$ and $h_{01} = 0$ into the embedding equation (2.1.5).

As we have already said the system (2.1.18)–(2.1.20) describes the classical motion of strings in curved spacetimes. This system is a system of the second-order non-linear coupled partial differential equations [88, 144, 145, 146, 155, 188, 193, 194, 197, 198]. The non-linearity of these equations gives a complication which leads to their non-integrability and possibly chaos [174]. It is well-known that various types of nonlinearities appear in Newtonian as well as relativistic systems and so they can deliver chaos. On the other hand, some types of non-linear equations can be integrable and their solutions are not chaotic. It seems that theory of relativity is an ideal to produce chaotic behaviour because their basic equations are highly non-linear. However, the problem is not as easy as one could think because most of the systems we study possess some symmetries which simplify the problem. This also refers to a single particle obeying both Newtonian and relativistic equations. Simply, a single particle which moves in the gravitational field of a source of gravity cannot move chaotically. However, two particles which form a 3-body system including the source can move in a chaotic way, though still not for all possible configurations.

Admission of extended objects such as strings gives another complication which, roughly, can be compared to the fact that now we have a many-body system which can obviously be chaotic on the classical level. An extended character of a string is reflected by the equations of motion which become a very complicated non-linear system from the very beginning. Thus, no wonder chaos can appear for classical evolution of strings around the simplest sources of gravity such as Schwarzschild black holes which was explicitly proven [87, 143]. This, for instance, means that there is no hope for making the full classification of the possible classical trajectories of strings in the Schwarzschild spacetime in analogy to Chandresekhar's classification for the point particles orbiting black holes [49]. However, in a similar way as for other types of non-linear sets of equations, there exist integrable configurations. The investigation of such explicit configurations can give an interesting insight into the problem of the general evolution of extended objects in various sources of gravity. Of course it is justified, provided we do not consider back-reaction of these extended objects onto the source field, i.e., if we consider test strings in analogy to test particles which do not "disturb" a gravitational field of a source.

The investigations of exact configurations can give an interesting insight into the problem. One useful example is when unstable periodic orbits (UPO) appear. Their emergence becomes a signal for a possible chaotic behaviour of the general system [60].

The task of this Chapter is to study some exact configurations for string moving in simple spacetimes of general relativity. Unfortunately, for tensile strings the main complication refers to the fact of their self-interaction reflected in the equations of motion by a non-zero value of their tension. However, as we have mentioned already, one is able to restrict oneself to studies of some simpler, but extended, configurations for which tension vanishes – null (tensionless) strings [133, 189]. We see that their equations of motion are just null geodesic equations of general relativity (2.1.18) appended by an additional "stringy" constraint (2.1.20). Many exact null string configurations in various curved spacetimes such as in Minkowski, Rindler, [132], Robertson-Walker [184] have been studied. One of the advantages of the null string approach is the fact that one may consider null strings as null approximation in various perturbative schemes for tensile strings [79, 156, 185, 196, 195, 216].

From now on our task will be to discuss the null and tensile string evolution in the black hole and cosmological spacetimes such as Schwarzschild, Kantowski-Sachs, Bianchi I and Bianchi IX. Since these calculations are purely classical, the string configurations we find can both be applied to fundamental strings and to well-known cosmic strings [203, 205].

2.2 Strings in Schwarzschild spacetime

The Schwarzschild spacetime is described by the metric

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \qquad (2.2.1)$$

and the set of spacetime coordinates is $X^\mu = (t, r, \theta, \phi)$ (see (2.1.15)). In the spacetime (2.2.1) from (2.1.18) we have

$$\ddot{t} - \lambda t'' + 2\frac{\frac{M}{r^2}}{1 - \frac{2M}{r}} \left(\dot{t}\dot{r} - \lambda t' r'\right) = 0, \qquad (2.2.2)$$

$$\ddot{r} - \lambda r'' - \frac{\frac{M}{r^2}}{1 - \frac{2M}{r}} \left(\dot{r}^2 - \lambda r'^2\right) + \frac{M}{r^2} \left(1 - \frac{2M}{r}\right) \left(\dot{t}^2 - \lambda t'^2\right) -$$

$$r \sin^2\theta \left(1 - \frac{2M}{r}\right) \left(\dot{\varphi}^2 - \lambda \varphi'^2\right) - r \left(1 - \frac{2M}{r}\right) \left(\dot{\theta}^2 - \lambda \theta'^2\right) = 0, \quad (2.2.3)$$

$$\ddot{\varphi} - \lambda \varphi'' + \frac{2}{r} \left(\dot{r}\dot{\varphi} - \lambda r'\varphi'\right) + 2\frac{\cos\theta}{\sin\theta} \left(\dot{\theta}\dot{\varphi} - \lambda \theta'\varphi'\right) = 0, \quad (2.2.4)$$

$$\ddot{\theta} - \lambda \theta'' + \frac{2}{r} \left(\dot{r}\dot{\theta} - \lambda r'\theta'\right) - \sin\theta\cos\theta \left(\dot{\varphi}^2 - \lambda \varphi'^2\right) = 0. \quad (2.2.5)$$

In the case of the null strings ($\lambda = 0$) the equations (2.2.2) and (2.2.4) can be easily integrated. The only difference from that of the general relativistic point particle case is that now the "constants of motion" must depend on the string coordinate σ, i.e.,

$$\dot{t} = \frac{E(\sigma)}{1 - \frac{2M}{r}}, \quad (2.2.6)$$

$$\dot{\varphi} = \frac{L(\sigma)}{r^2 \sin^2\theta}. \quad (2.2.7)$$

Combining (2.2.5) with (2.2.6) and (2.2.7) we obtain for $\lambda = 0$

$$r^4 \sin^2\theta \ddot{\theta} + 2r^3 \dot{r} \sin^2\theta \dot{\theta} - L^2(\sigma) \frac{\cos\theta}{\sin\theta} = 0, \quad (2.2.8)$$

which can be integrated in a standard way

$$r^4 \sin^2\theta \dot{\theta}^2 = -L^2(\sigma) \cos^2\theta + K(\sigma) \sin^2\theta, \quad (2.2.9)$$

and the non-negative function $K(\sigma)$ generalizes Carter's 'fourth constant' of motion (see for instance [171]). The standard potential equation for the radial coordinate is then obtained by integrating (2.2.3)) in the case $\lambda = 0$

$$\dot{r}^2 + V(r) = 0, \quad (2.2.10)$$

where

$$V(r) = -E^2(\sigma) + \frac{1}{r^2} \left(1 - \frac{2M}{r}\right) \left[L^2(\sigma) + K(\sigma)\right], \quad (2.2.11)$$

and the constraint (2.1.19) has been taken into account. As we can easily see $L(\sigma)$ refers directly to the coordinate φ while $K(\sigma)$ refers directly to the coordinate θ. One can also define the generalized impact parameter as [49]

$$D(\sigma) \equiv \frac{\sqrt{L^2(\sigma) + K(\sigma)}}{E(\sigma)}. \tag{2.2.12}$$

It should be noticed that if there was not the "constant of motion" $K(\sigma)$, the only solution of (2.2.9) with $L \neq 0$ would be given by $\theta = const. = \pi/2$. As it is very well known in the case of point particles one can of course put $K = 0$ without loss of generality. However, because of the fact that the string is an extended object, this is not the case here, and the explicit dependence on the string coordinate σ must be given. In particular, the null string does *not* in general move in a plane through the origin, although each individual point of the string actually does.

The constraint (2.1.20) now takes the form

$$E(\sigma)t' - \frac{\dot{r}}{1 - \frac{2M}{r}} r' - L(\sigma)\varphi' - r^2\dot{\theta}\theta' \sin^2\theta = 0, \tag{2.2.13}$$

with $\dot{r}$ and $\dot{\theta}$ given by (2.2.9)–(2.2.10). Equation (2.2.13) means that we have a constraint on the functions $E(\sigma)$, $L(\sigma)$ and $K(\sigma)$.

Finally, we notice that the invariant string size (the length of the string) (2.1.17) for the null string is given by the integral of

$$S(\tau,\sigma) = \left[-\left(1 - \frac{1}{2M}\right) t'^2 + \left(1 - \frac{1}{2M}\right)^{-1} r'^2 + r^2\theta'^2 + r^2 \sin^2\theta \varphi'^2 \right]^{1/2}. \tag{2.2.14}$$

2.2.1 Circular null strings

As a first example of exact solutions, we consider the circular ansatz for a null string in the Schwarzschild spacetime:

$$t = t(\tau), \quad r = r(\tau), \quad \theta = \theta(\tau), \quad \varphi = \sigma. \tag{2.2.15}$$

Inserting (2.2.15) into (2.2.6)–(2.2.10) we have

$$\dot{t} = \frac{E(\sigma)}{1-\frac{2M}{r}}, \tag{2.2.16}$$

$$\dot{r}^2 = E^2(\sigma) - \frac{K(\sigma)}{r^2}\left(1-\frac{2M}{r}\right), \tag{2.2.17}$$

$$r^4\dot{\theta}^2 = K(\sigma), \tag{2.2.18}$$

and we have also used the constraint (2.2.13) which implies that L is equal to zero. It is also clear from (2.2.16)–(2.2.18) that E and K must be *constants* (independent of σ) in this case. This is not the case in general, of course.

The simplest solutions come if we put $K = 0$, i.e.,

$$\theta = const., \tag{2.2.19}$$

$$r - r_0 + 2M \ln\frac{r-2M}{r_0-2M} = \pm(t-t_0), \tag{2.2.20}$$

and they describe the "cone strings" which start with finite size and then sweep out the light cones $\theta =$ const. (Fig. 2.1) These strings play similar role as the radial null geodesics in General Relativity. Notice that the extended nature of the string means that configurations corresponding to different values of the constant polar angle θ are physically different: for $\theta = \pi/2$, the string is in a plane through the origin while for any other value of θ it is not. In fact, for $\theta = \pi/2$, the string winds around the black hole in the equatorial plane while for $\theta \neq \pi/2$, it is on a parallel plane moving perpendicular to the equatorial plane. In contrast, a point particle is always in a plane through the origin. This illustrates the fact that although the null string equations are similar to the massless geodesic equations, in particular when E and K are constants, the physical interpretation of the solutions is completely different.

For the sake of comparison we mention that in Minkowski spacetime ($M = 0$) the logarithmic term vanishes, giving:

$$\theta = const., \tag{2.2.21}$$

$$r - r_0 = \pm(t-t_0). \tag{2.2.22}$$

The "cone strings" also appear in the anti-de-Sitter spacetime since there

$$\theta = const., \tag{2.2.23}$$
$$r - r_0 = \pm\frac{1}{H}\tan H\,(t - t_0), \tag{2.2.24}$$

with $H = \text{const.} = \sqrt{-\Lambda/3}$, where Λ is the cosmological constant. Such "cone strings" can also easily be constructed in other static spherically symmetric spacetimes, such as, for instance, the Reissner-Nordström spacetime [49, 118].

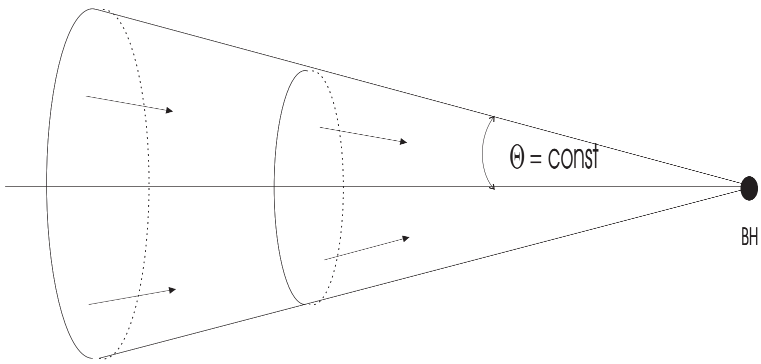

Figure 2.1: The evolution of a "cone" string in a black hole (BH) spacetime

Coming back to the equations (2.2.16)–(2.2.18) for arbitrary K, we first notice that (2.2.17) is exactly equivalent to the equation for photons moving in the equatorial plane with non-vanishing angular momentum ($L \neq 0,\ \theta = \pi/2 \ \Rightarrow\ K = 0$). This, of course, just reflects the fact that point particles always move in a plane through the origin. However, since a string (even a null string) is an extended object, the physical interpretation of the solutions to (2.2.17) is completely different from that of point particle solutions. In particular, none of the string solutions we will obtain here are propagating in a plane through the origin. But the qualitative and quantitative picture for the string solutions can still be extracted from the well-known results for point particles (see for instance [49, 171]). It is possible because in the assumed ansatz (2.2.15), all the spacetime coordinates are just functions of one of the two string coordinates, and we can derive the equation which relates the coordinates r and θ from (2.2.16)–(2.2.18) in a similar way as one usually does for the massless particles [49], although now, instead of φ we have θ, and instead of L we have $\sqrt{K}$, i.e.,

$$\begin{aligned} \left(\frac{du}{d\theta}\right)^2 &= 2M\left(u+\frac{1}{6M}\right)\left(u-\frac{1}{3M}\right)^2+\frac{1}{M^2}\left(\frac{M^2}{D^2}-\frac{1}{27}\right) \\ &= 2Mu^3-u^2+\frac{1}{D^2}, \end{aligned} \tag{2.2.25}$$

where $u = 1/r$ and D is defined by (2.2.12). From this equation we can immediately derive the trajectories of the null strings similarly as in [49]. In our discussion we just use directly (2.2.17) looking for the turning points ($\dot{r} = 0$) of it, fulfilling

$$\frac{K}{4M^2E^2} = \frac{(r/2M)^2}{1-(r/2M)^{-1}}. \tag{2.2.26}$$

Equation (2.2.26) has two solutions outside the horizon provided

$$\frac{K}{M^2E^2} = \frac{D^2}{M^2} > 27, \tag{2.2.27}$$

one solution in the case of equality, and no solutions otherwise. We first consider the case for which $K < 27M^2E^2$ ($D < 3\sqrt{3}M$) and a circular string is incoming from spatial infinity (say) $\theta = 0$, $r = \infty$. The plane of the string is always parallel to the equatorial plane and the string approaches the south pole of the black hole, and since there is no turning point in this case, the string will eventually fall onto the black hole. If $K > 27M^2E^2$ ($D > 3\sqrt{3}M$), the string again approaches the south pole of the black hole, but in this case it will scatter off and escape back towards infinity. Notice that in both cases the string can make a number of turns, moving vertically from the south pole to the north pole and back again and so on (during which it actually collapses ($r\sin\theta = 0$) infinitely many times), but always with the plane of the string parallel to the equatorial plane of Schwarzschild geometry. Similarly, one can consider strings starting very close, but outside the horizon with increasing $r(\tau)$. If $K < 27M^2E^2$, the string escapes to infinity while if $K > 27M^2E^2$, it hits the barrier and fall back onto the black hole. In the next subsection we will consider a limiting case of this kind of dynamics.

2.2.2 Null strings on the photon sphere

In the special case when the impact parameter $D = 3\sqrt{3}M$, Eq. (2.2.25) (with the circular ansatz (2.2.15) valid) factorizes and the simplest solution for the constant radial coordinate $r = 1/u = 3M$ comes immediately. If $r = 3M$, then we conclude from (2.2.25)) and (2.2.16) that

$$t(\tau) = 3E\tau, \qquad D^2 = 27M^2. \tag{2.2.28}$$

Then, one is able to integrate (2.2.18) to give

$$\theta = \pm\frac{E\tau}{\sqrt{3}M} + \theta_0, \tag{2.2.29}$$

and $\theta_0 = \text{const}$. Finally, we conclude that a circular string solution is described explicitly by [77]

$$t = 3E\tau, \quad r = const. = 3M, \quad \theta = \pm\frac{E\tau}{\sqrt{3}M} + \theta_0, \quad \varphi = \sigma, \tag{2.2.30}$$

which means that the string may move vertically from the south pole to the north pole and back again and so on around the photon sphere ($r = 3M$) [49, 171]. Notice that the factor $3E$ in equation (2.2.28) can be scaled away.

The invariant string size (2.1.17) in the case of the string solution (2.2.30) is simply

$$S = 6\pi M \sin\left(\pm\frac{E\tau}{\sqrt{3}M} + \theta_0\right). \tag{2.2.31}$$

Notice that the string time coordinate τ extends to infinity while the angle θ in Schwarzschild spacetime has values limited to $0 < \theta < \pi$. This is not a problem since the function $\sin(\theta)$ is periodic and the invariant string size (2.2.31) changes periodically from 0 to $6\pi M$. The string solution (2.2.30) with the constant radial coordinate $r = 3M$ is, except the solution for a string on the event horizon

$$t = \tau, \quad r = const. = 2M, \quad \theta = const., \quad \varphi = \sigma, \tag{2.2.32}$$

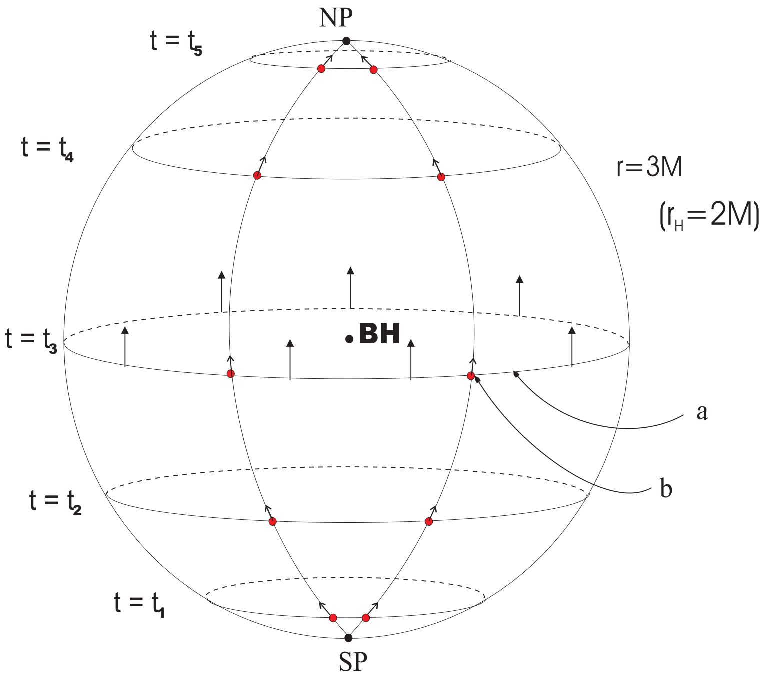

Figure 2.2: The evolution of a string on the photon sphere in Schwarzschild spacetime: a) a string at the moment of passing the equatorial plane, b) a point of the string moving all the time in the plane through the origin of coordinates $r = 0$. BH – black hole singularity (here spacelike), r = 3M – the radius of the photon sphere, $r_H = 2M$ – the event horizon

the only string solution with constant r [132]. It is important that the solution $r = 2M$ (this is due to the fact that $r = 2M$ is an isotropic (null) surface) is stationary while the solution $r = 3M$ is highly dynamical. It must be stressed, however, that the solution $r = 3M$ is unstable. The situation is similar to the case of a photon on a circular orbit at $r = 3M$ [49, 171] (see Fig. 2.2) and it means that there exist two asymptotic solutions in addition to the one given by (2.2.30), one of which describes an incoming string approaching the photon sphere from infinity ($r = \infty$), then spirals around it infinitely many times, and the other one, an outgoing string, starting from somewhere close to (but outside) the horizon approaching the photon sphere also by spiraling around it infinitely many times. The exact expressions for such trajectories have already been given [49]. For the string approaching the photon sphere from infinity (the orbit of 'the first kind' as called in [49]) we have from (2.2.16)–(2.2.18) [77]

$$\frac{1}{r} = -\frac{1}{6M} + \frac{1}{2M}\tanh^2\left(\frac{1}{2}(\theta+\theta_0)\right), \tag{2.2.33}$$

$$\frac{dt}{d\tau} = \frac{E}{\frac{4}{3} - \tanh^2\left(\frac{1}{2}(\theta+\theta_0)\right)}, \tag{2.2.34}$$

$$\frac{d\theta}{d\tau} = \pm\frac{E}{4\sqrt{3}M}\left[-1 + 3\tanh^2\left(\frac{1}{2}(\theta+\theta_0)\right)\right]^2, \tag{2.2.35}$$

and for the outgoing string approaching the photon sphere (the orbit of 'the second kind') [77]

$$\frac{1}{r} = \frac{1}{3M} + \frac{2z}{M(z-1)^2}, \tag{2.2.36}$$

$$\frac{dt}{d\tau} = \frac{E}{\frac{1}{3} - \tan^2\left(2\arctan(\sqrt{z})\right)}, \tag{2.2.37}$$

$$\frac{dz}{d\tau} = \pm\frac{E}{\sqrt{3}M}\,\frac{z\left[z^2+4z+1\right]^2}{(z-1)^4}, \tag{2.2.38}$$

where $z = \exp\theta$ and $\theta_0 = \text{const}$. The exact solutions of these equations are given in terms of elliptic functions and we will not discuss them further on since their physical meaning is clear from the qualitative considerations given above and at the end of subsection 2.2.1. One can easily see from (2.2.33)–(2.2.35) that in the limit $\theta \to \infty$, $r = 3M$, $t = 3E\tau$ and $d\theta/d\tau = \pm E/\sqrt{3}M$. Similarly in (2.2.36)–(2.2.38) for $\theta \to \infty$, $r = 3M$, $t = 3E\tau$ and $d\theta/d\tau = \pm E/\sqrt{3}M$, thus in both cases we obtain the limiting case (2.2.30) as we should.

2.2.3 Circular tensile strings

In this subsection we briefly consider the case of tensile strings and start with the circular ansatz of subsection 2.2.1 given by Eq. (2.2.15). The equations of motion (2.2.2)–(2.2.5) are given by

$$\dot{t} - \frac{E}{1-\frac{2M}{r}} = 0, \tag{2.2.39}$$

$$\ddot{r} - \frac{\frac{M}{r^2}}{1-\frac{2M}{r}}\dot{r}^2 + \frac{M}{r^2}\left(1-\frac{2M}{r}\right)\dot{t}^2 + r\sin^2\theta\left(1-\frac{2M}{r}\right)$$
$$- r\left(1-\frac{2M}{r}\right)\dot{\theta}^2 = 0, \qquad (2.2.40)$$

$$\ddot{\theta} + 2\frac{\dot{r}}{r}\dot{\theta} + \sin\theta\cos\theta = 0, \qquad (2.2.41)$$

and the constraint (2.1.20) is automatically fulfilled, while the constraint (2.1.19) gives

$$\frac{E^2 - \dot{r}^2}{\left(1-\frac{2M}{r}\right)} - r^2\dot{\theta}^2 - r^2\sin^2\theta = 0. \qquad (2.2.42)$$

These equations can be easily integrated in the equatorial plane and the solutions were discussed in [144, 145, 193]. The special solutions of (2.2.39)–(2.2.41) outside the equatorial plane are the ones with constant radial coordinate $r=$ const. However, as one can easily check by simple substitution $\dot{r}=0$ in (2.2.39)–(2.2.41) and then differentiation of (2.2.40) with respect to τ, this results in contradiction with (2.2.41). This means that tensile strings with a constant value of the radial coordinate r do not exist at all. This is a very big difference from the point particles since for massive particles there are circular orbits which are stable for $r > 6M$ and unstable for $r < 6M$. It seems that it is impossible to keep such a symmetric and stationary configuration because of the self-interaction of the tensile strings.

A second first integral (besides (2.2.42)) of the system (2.2.39)–(2.2.41), which would guarantee its full integrability, is not known. In fact, numerical investigations proved [87] that no such integral exists, i.e., the system is chaotic. These numerical investigations showed that there are essentially three possible evolution schemes for the axisymmetric tensile string in the Schwarzschild spacetime [143, 193]: (1) the string simply passes the horizon and falls onto the black hole. (2) the string passes by the black hole, but part of its translational energy is transformed into oscillatory energy. (3) the string is "trapped" jumping chaotically around the black hole for a certain amount of time before it either falls into the black hole or escapes towards infinity.

Notice that this is qualitatively the same kinds of dynamics that we found for the null strings using exact analytical methods of subsections 2.2.1 and 2.2.2. This suggests that by fine-tuning of the initial conditions, it should actually be possible to have (tensile) string solutions jumping around the black hole forever, somewhat similar to the null string on the photon sphere as discussed in subsection 2.2.2, but with the radial coordinate not exactly constant. In the present case of tensile strings it is, however, not known whether such solutions would be stable under small perturbations, and their existence might demand *infinite* fine-tuning of the initial conditions. We finally notice that in the case of electrically charged strings, such solutions do actually exist [143]; their stability being guaranteed by a Coulomb barrier for a small radius and a tension barrier for large radius.

After our analysis of the evolution of strings in the above cases we now come to the following discussion which compares the behaviour of classical null strings and tensile strings to the behaviour of massless and massive point particles in curved spacetimes.

A classical massive point particle in a curved spacetime experiences only the interaction with the gravitational field. It means that taking the limit of zero mass changes the world-line of the particle smoothly since photons obeys the same rule and they also interact with the gravitational field only. Thus from the point of view of world-lines only, the limit of zero mass is relatively smooth. For strings the situation is somewhat more complicated. What distinguishes the strings from the point particles in such situations is that the strings not only interact with the gravitational field, but also selfinteract due to the tension. This makes an important physical difference with what we have for the point particles in the sense that now taking the limit of zero tension is quite *dramatic*. It is because in this limit the selfinteraction of strings totally vanishes, while the interaction with the gravitational field changes smoothly (as for point particles).

Although each individual point along the null string follows a null geodesic, the null string as a whole may experience highly non-trivial dynamics. The situation is qualitatively similar to that of congruences in general relativity, that is, "bundles of rays" – each ray in the bundle is just following a geodesic, but the propagation of the bundle as a whole can be highly non-trivial due to tidal forces as described by the Raychaudhuri equation [171, 186].

It is thus obvious that the null-particles (like photons) have something to do with massive particles. The general question is then, whether the null strings actually have anything to do with the tensile strings. In subsection 2.2.4 we develop an idea (see [156, 195, 196, 216]) that the tensile string equations of motion may be expanded perturbatively in inverse string tension parameter α' (or a parameter related to it by rescaling) having the null string equations of motion as the zeroth order approximation.

On the other hand, it is certainly meaningful and interesting to consider null strings by themselves. The dynamics of null strings in a curved spacetime is, however, very simply obtained if the dynamics of point particles is known. Knowing this, it is essentially a question of interpreting the well-known point particle results within the framework of extended objects. Referring to our calculations one can distinguish two different physical situations. These are when the constraint (2.1.20) is or is not automatically fulfilled by the ansatz. If it is automatically fulfilled, the evolution of a string is almost trivial in the sense that each point of the string follows a null geodesic (a trajectory of a massless particle) without any correlations with the rest of the string. Then, the string motion reduces to the motion of a collection of massless point particles moving quite *independently*. On the other hand, if the constraint (2.1.20) is not automatically fulfilled by the ansatz then there are some non-trivial correlations between the different points of the string, the nature of which are purely 'stringy'. It appears that the 'stringy' nature is absent in any axially symmetric spacetime for the circular ansatz, thus to look for more complicated dynamics one should consider either some less symmetric background spacetimes or some different string shapes.

Similar problems as discussed here can be discussed for the null p-branes in curved spacetimes [185] which generalize strings to higher-dimensional objects.

2.2.4 String dynamics in perturbative approach

From the previous subsections we have learned that the exact solutions in Schwarzschild spacetime are possible for the null strings. However, the important physical information about tensile string dynamics can be obtained from studying the approximate solutions of its equations of motion as it was proposed by de Vega and Sanchez [197, 198] and by Roschupkin and Zheltukhin [185, 216]. As for the latter the perturbative scheme was based on the assumption of a small value of a rescaled string tension parameter. Some other proposals were also made in [156, 195, 196].

Our main task in this subsection is to apply the expansion scheme as proposed by Zheltukhin in [216] for studying string dynamics in Schwarzschild spacetime and compare the results with a qualitative picture given in our subsection 2.2.2. For the sake of that the solution for a null string moving along the photon sphere of the Schwarzschild spacetime (2.2.30) is applied.

The basic idea is to use a generalization of the action given for massless point particles [110]. We apply a perturbative parameter (2.1.10) onto strings. It was shown in [216] that for small $\varepsilon \ll 1$ one can introduce a macroscopic 'slow' worldsheet time parameter

$$\bar{T} = \varepsilon\tau, \tag{2.2.43}$$

where τ is the proper string time parameter. On the scale $\bar{T}$ the string oscillations can be considered as perturbations with respect to the translational motion of the string points and described in the form of asymptotic expansion

$$X^\mu(\bar{T},\sigma) = \varphi^\mu(\bar{T}) + \varepsilon\chi^\mu(\bar{T},\sigma) + \varepsilon^2\zeta^\mu(\bar{T},\sigma) + \ldots, \tag{2.2.44}$$

with σ being a spacelike worldsheet string coordinate and $\mu,\nu,\rho,\kappa = 0,1,2,3$. After introducing the expansion (2.2.44) the perturbative equations of motion and constraints (2.1.18)–(2.1.20) in the first approximation have the following form [185]

$$\left(\mathcal{D}^2_{\bar{T}} - \partial^2_\sigma\right)\chi^\mu + R^\mu_{\nu\rho\kappa}(\varphi)\varphi^\nu_{,\bar{T}}\varphi^\rho_{,\bar{T}}\chi^\kappa \quad = \quad 0,$$

$$\begin{aligned} \left(\varphi_{\mu,\bar{T}}\mathcal{D}_{\bar{T}}\chi^{\mu}\right) &= 0, \qquad (2.2.45)\\ \left(\varphi_{\mu,\bar{T}}\chi^{\mu}\right) &= 0, \end{aligned}$$

where $\mathcal{D}_{\bar{T}}\chi^{\mu} = \chi^{\mu}_{,\bar{T}} + \varphi^{\nu}_{,\bar{T}}\Gamma^{\mu}_{\nu\kappa}(\varphi)\chi^{\kappa}$, $R^{\mu}_{\nu\rho\kappa}$ is the Riemann tensor and $(\ldots)_{,\bar{T}} = \partial/\partial\bar{T}$. First of the Eqs. (2.2.45) is of the form of the geodesic deviation equation with an additional term $\partial^2_{\sigma}\chi^{\mu}$ describing the elastic string force. The zeroth order equations for $\varphi^{\mu}(\bar{T})$ are geodesic equations for a massless particle in a given curved space, i.e.,

$$\begin{aligned} \mathcal{D}_{\bar{T}}\varphi^{\mu}_{,\bar{T}} &= 0,\\ \left(\varphi^{\mu}_{,\bar{T}}\varphi_{,\bar{T}\mu}\right) &= 0. \qquad (2.2.46) \end{aligned}$$

Now we want to use this perturbative scheme for the discussion of string dynamics in Schwarzschild spacetime with the line element given by (2.2.1). We assume M is the Schwarzschild mass with the dimension of (length)$^{-1}$, $\hbar = c = 1$, and introduce explicitly the Newton constant G into the metric (2.2.1) simply by changing M into GM and letting G to have the dimension of (length)^2.

Our suggestion is to choose $\gamma = M^{-2}$ in (2.1.10) to play the role of the constant γ in the rescaled tension parameter so that

$$\varepsilon = \frac{M^{-2}}{\alpha'}, \qquad (2.2.47)$$

and the perturbative approach can be applied for the case when $M^{-2} \ll \alpha'$.

In order to solve the perturbative equations (2.2.45) for the first approximation functions χ^{μ} we need the solution for the zero-order approximation functions φ^{μ}. The general solution of the geodesic equations (2.2.46) for a massless particle in Schwarzschild spacetime is well-known [49]. Here we want to apply a particular form of the solution of the geodesic motion which describes a massless particle moving on the photon sphere in Schwarzschild spacetime given by (2.2.30)

$$\varphi^0(\bar{T}) = 3E\bar{T}, \quad \varphi^1(\bar{T}) = 3GM, \quad \varphi^2(\bar{T}) = \pm\frac{E\bar{T}}{\sqrt{3}GM} + \theta_0, \quad \varphi^3(\bar{T}) = \phi_0, \tag{2.2.48}$$

with E, θ_0, ϕ_0 constant. It is important that (2.2.48) automatically satisfies the constraint (2.2.46).

Substitution of (2.2.48) into the equations for the first approximation functions (2.2.45) and making use of the background metric components (2.2.1) we get the equations of motion for χ^μ

$$\begin{aligned}
\chi^0_{,\bar{T}\bar{T}} - \chi^0_{,\sigma\sigma} + 2\frac{E}{GM}\chi^1_{,\bar{T}} &= 0,\\
\chi^1_{,\bar{T}\bar{T}} - \chi^1_{,\sigma\sigma} + 2\left[\frac{E}{9M}\chi^0_{,\bar{T}} \mp \frac{E}{\sqrt{3}}\chi^2_{,\bar{T}} - \frac{E^2}{6G^2M^2}\chi^1\right] &= 0, \qquad (2.2.49)\\
\chi^2_{,\bar{T}\bar{T}} - \chi^2_{,\sigma\sigma} \pm \frac{2E}{3\sqrt{3}G^2M^2}\chi^1_{,\bar{T}} &= 0,\\
\chi^3_{,\bar{T}\bar{T}} - \chi^3_{,\sigma\sigma} \pm \frac{E}{\sqrt{3}GM}\cot\left(\pm\frac{E}{\sqrt{3}GM}\bar{T} + \theta_0\right)\chi^3_{,\bar{T}} &= 0,
\end{aligned}$$

and the constraints are [185]

$$\begin{aligned}
\chi^0_{,\sigma} \mp 3\sqrt{3}GM\chi^2_{,\sigma} &= 0, \qquad (2.2.50)\\
\chi^0_{,\bar{T}} \mp 3\sqrt{3}GM\chi^2_{,\bar{T}} &= 0.
\end{aligned}$$

The constraints (2.2.50) can be integrated to give only one condition

$$\chi^0 = \mp 3\sqrt{3}GM\chi^2, \tag{2.2.51}$$

and the set of equations (2.2.49) reads

$$\chi^0_{,\bar{T}\bar{T}} - \chi^0_{,\sigma\sigma} + 2\frac{E}{GM}\chi^1_{,\bar{T}} = 0, \tag{2.2.52}$$

$$\chi^1_{,\bar{T}\bar{T}} - \chi^1_{,\sigma\sigma} - \frac{E^2}{3G^2M^2}\chi^1 = 0, \tag{2.2.53}$$

$$\chi^2_{,\bar{T}\bar{T}} - \chi^2_{,\sigma\sigma} \mp 2\frac{E}{3\sqrt{3}G^2M^2}\chi^1_{,\bar{T}} = 0, \qquad (2.2.54)$$

$$\chi^3_{,\bar{T}\bar{T}} - \chi^3_{,\sigma\sigma} \pm \frac{E}{\sqrt{3}GM}\cot\left(\pm\frac{E}{\sqrt{3}GM}\bar{T} + \theta_0\right)\chi^3_{,\bar{T}} = 0. \qquad (2.2.55)$$

The simple solution of (2.2.53) is given by

$$\chi^1 = Ae^{i(\omega\bar{T} - k\sigma)}, \qquad (2.2.56)$$

where $A =$ const. and one has a dispersion relation of the form

$$\omega(k) = \sqrt{k^2 - \frac{E^2}{3G^2M^2}}. \qquad (2.2.57)$$

We can also find a separable solution of (2.2.55) of the form

$$\chi^3(\bar{T}, \sigma) = f(\bar{T})g(\sigma), \qquad (2.2.58)$$

which reads (w is a separation constant)

$$g(\sigma) = e^{iw\sigma}, \qquad (2.2.59)$$

and $f(\bar{T})$ is a solution of the Legendre equation

$$(\zeta^2 - 1)f_{,\zeta\zeta} + 2\zeta f_{,\zeta} - m(m+1)f = 0, \qquad (2.2.60)$$

with ($\mid \zeta \mid < 1$)

$$\zeta = \cos\left(\pm\frac{E}{\sqrt{3}GM}\bar{T} + \theta_0\right), \qquad (2.2.61)$$

$$m(m+1) = \frac{w^2E^2}{3G^2M^2}. \qquad (2.2.62)$$

The easiest solution one could get is for χ^1 being independent of time $\bar{T}$ in (2.2.49).

After inspecting (2.2.50) we can easily notice that (2.2.52) results in three two-dimensional wave equations for χ^0, χ^2 and χ^3 which refer to the perturbations in the components t, θ and ϕ of the metric (2.2.1), respectively. The solutions for these components are

$$\begin{aligned}\chi^0 &= \sum_{k=-\infty}^{\infty}\Big(\alpha_k^0 e^{ik(\sigma-\bar{T})}+\beta_k^0 e^{-ik(\sigma-\bar{T})}\Big),\\ \chi^2 &= \sum_{k=-\infty}^{\infty}\Big(\alpha_k^2 e^{ik(\sigma-\bar{T})}+\beta_k^2 e^{-ik(\sigma-\bar{T})}\Big),\\ \chi^3 &= \sum_{k=-\infty}^{\infty}\Big(\alpha_k^3 e^{ik(\sigma-\bar{T})}+\beta_k^3 e^{-ik(\sigma-\bar{T})}\Big)\end{aligned} \tag{2.2.63}$$

which describe the small string oscillations with frequencies $k = 1, 2\ldots$ on the macroscopic scale $\bar{T}$. The emergence of these oscillations is a direct consequence of admitting the non-zero (but very small) string tension. These oscillations are the oscillations which take place on the surface of the photon sphere and they do not lead to any deformations of this sphere. On the other hand, the equation for the radial correction χ^1 in (2.2.52) has the following solution

$$\chi^1 = A^1 \sin\left(\frac{E}{\sqrt{3}GM}\sigma\right) + B^1 \cos\left(\frac{E}{\sqrt{3}GM}\sigma\right), \tag{2.2.64}$$

with A^1, B^1 constant. The solution (2.2.64) must obey the periodicity condition

$$\chi^1(0) = \chi^1(2\pi). \tag{2.2.65}$$

As a consequence of (2.2.65) we get the 'quantization' of the parameter E

$$E = \sqrt{3}GMn \qquad (n = 0, \pm 1, \pm 2, \ldots), \tag{2.2.66}$$

and the final solution for χ^1 is

$$\chi^1 = A^1 \sin n\sigma + B^1 \cos n\sigma. \tag{2.2.67}$$

From (2.2.67) we can conclude that in contradiction to the case of oscillations for the components χ^0, χ^2 and χ^3 as described by (2.2.63), the solution (2.2.67) corresponds to a small static deformation of the photon sphere $r = 3GM + \varepsilon(A^1 \sin n\sigma + B^1 \cos n\sigma)$ for $\varepsilon \ll 1$.

The successful application of the perturbative scheme of [216] to the qualitative discussion of our subsection 2.2.3 for the particular zero-order solution (a string on the photon sphere) in Schwarzschild spacetime appears to give hope that the various exact solutions for the particle motion in Schwarzschild spacetime [49] can be considered as zero-order approximations for the perturbative description of the tensile string dynamics in this spacetime. The scheme was also applied to some other spacetimes [185].

2.3 Strings in cosmological spacetimes

In this Section we extend the discussion of the tensile and null string evolution to the homogeneous, but anisotropic, spacetimes of Kantowski-Sachs, Bianchi types I and IX. In fact, Kantowski-Sachs solutions with negative and zero curvature are just axisymmetric Bianchi type I and III universes. This means that only positive curvature Kantowski-Sachs models are different from Bianchi type universes [186].

2.3.1 Strings in Kantowski-Sachs spacetime

Let us first consider the string equations of motion in homogeneous Kantowski-Sachs spacetimes. The validity of the Kantowski-Sachs spacetime as being consistent string vacuum (solution to the β-function equations to the lowest order in α') is discussed in section 3.3. Here we just concentrate on the motion of a test string in Kantowski-Sachs backgrounds.

The Kantowski-Sachs spacetime is given by the metric [131]

$$ds^2 = dt^2 - A^2(t)dr^2 - B^2(t)d\Omega^2_{\mathcal{K}}, \tag{2.3.1}$$

where the "angular" metric is

$$d\Omega^2_{\mathcal{K}} \quad = \quad d\theta^2 + S^2(\theta)d\varphi^2, \tag{2.3.2}$$

$$S(\theta) \quad = \quad \begin{cases} \sin\theta & \text{for} \quad \mathcal{K} = +1, \\ \theta & \text{for} \quad \mathcal{K} = 0, \\ \sinh\theta & \text{for} \quad \mathcal{K} = -1, \end{cases} \tag{2.3.3}$$

and A, B are the expansion scale factors and $\mathcal{K}$ is the curvature index. Here, $r \in]-\infty, \infty[$, while the range of t depends on the particular cosmology. For $\mathcal{K} = +1$ the coordinates θ and φ describe, as usual, the angles on the 2-sphere. Only $\mathcal{K} = +1$ models fall outside the Bianchi classification, but usually one refers to all three curvature models as Kantowski-Sachs universes. In this book, we mainly consider $\mathcal{K} = +1$ models.

As a first example of a string configuration, we apply the following string ansatz

$$X^0 = t(\tau), \quad X^1 = r(\tau), \quad X^2 = \theta(\tau), \quad X^3 = \varphi = \sigma, \tag{2.3.4}$$

which describes a circular string winding around the 2-sphere. The functions $(t(\tau), r(\tau),$
$\theta(\tau))$, which describe the dynamics of the string, are to be determined from the equations of motion.

For the metric (2.3.1), we start with the string equations of motion (2.1.18) and constraints (2.1.19)–(2.1.20), which now reduce to [78]

$$\ddot{t} + AA_{,t}\dot{r}^2 + BB_{,t}\dot{\theta}^2 - \lambda BB_{,t}\sin^2\theta = 0, \tag{2.3.5}$$

$$\ddot{r} + 2\frac{A_{,t}}{A}\dot{t}\dot{r} = 0, \tag{2.3.6}$$

$$\ddot{\theta} + 2\frac{B_{,t}}{B}\dot{t}\dot{\theta} + \lambda\sin\theta\cos\theta = 0, \tag{2.3.7}$$

$$\dot{t}^2 - A^2\dot{r}^2 - B^2\dot{\theta}^2 - \lambda B^2\sin^2\theta = 0. \tag{2.3.8}$$

The Eq. (2.3.6) can be easily integrated to give

$$\dot{r} = \frac{dr}{d\tau} = \frac{k}{A^2}, \tag{2.3.9}$$

with $k = \text{const}$. The other equations, in general, cannot be integrated, thus we must either consider special Kantowski-Sachs spacetimes or make further restrictions for the ansatz. Notice, however, that for both cases $\lambda = 0, 1$, the invariant string size is given by

$$S(\tau) = 2\pi|B(\tau)\sin\theta(\tau)|. \tag{2.3.10}$$

A. Tensile strings

For the tensile strings, $\lambda = 1$, and (2.3.7) is fulfilled automatically under the assumption that $\theta = \text{const.} = \pi/2$ in (2.3.4). In the rest of this subsection, we restrict ourselves to this case. Then, after inserting (2.3.9) into (2.3.8), we obtain

$$\dot{t}^2 = \frac{k^2}{A^2} + B^2, \tag{2.3.11}$$

or, explicitly in terms of the string time coordinate,

$$\tau(t) = \int^t \frac{|A|\, dt}{\sqrt{k^2 + A^2 B^2}}, \tag{2.3.12}$$

while from (2.3.9) we get

$$r(t) = k \int^t \frac{dt}{|A|\sqrt{k^2 + A^2 B^2}}. \tag{2.3.13}$$

Notice also that (2.3.5) is automatically fulfilled now.

a) Λ-term solutions

First we refer to one of the simplest solutions given for the scale factors, which is the $\mathcal{K} = +1$ Kantowski-Sachs universe with only the cosmological term Λ [112]. These are:

$$A(t) = H_0^{-1} \sinh H_0 t, \tag{2.3.14}$$

$$B(t) = H_0^{-1} \cosh H_0 t, \tag{2.3.15}$$

with $H_0 = \sqrt{\Lambda/3} = \text{const.}$ [118], and we consider only the expanding phase ($t \geq 0$) of the universe. After inserting (2.3.14)–(2.3.15) into (2.3.12) we have (choosing boundary conditions such that $t(0) = 0$)

$$\tau(t) = H_0 \int_0^t \frac{\sinh (H_0 t)}{\sqrt{k^2 H_0^4 + \sinh^2 (H_0 t) \cosh^2 (H_0 t)}}\, dt, \tag{2.3.16}$$

and from (2.3.13) we obtain

$$r(t) = k H_0^3 \int_{t_0}^t \frac{dt}{\sinh (H_0 t)\sqrt{k^2 H_0^4 + \sinh^2 (H_0 t) \cosh^2 (H_0 t)}}. \tag{2.3.17}$$

By inverting (2.3.16), we find $t(\tau)$, and Eq. (2.3.17) then gives explicitly $r(\tau)$. The Eqs. (2.3.16)–(2.3.17) can be transformed to the form of the standard elliptic integrals. For instance, by making the substitution $\cosh(H_0 t) = \sqrt{z}$, Eq. (2.3.16) becomes

$$\tau(z) = \frac{1}{2}\int_1^z \frac{dz}{\sqrt{z(z-z_1)(z-z_2)}}, \tag{2.3.18}$$

with

$$z_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{1 - 4k^2 H_0^4}\right), \tag{2.3.19}$$

and can thus be evaluated explicitly eventually yielding $t(\tau)$. However, the detailed form will not be important here. The invariant string size in this case is simply

$$S(\tau) = 2\pi H_0^{-1} \cosh(H_0 t(\tau)), \tag{2.3.20}$$

i.e., the string size follows the expansion of the Universe. This is easily understood, since the string is simply winding around the equatorial plane of the 2-sphere.

b) Time-symmetric stiff-fluid solutions

The solutions for the scale factors for the time-symmetric $\mathcal{K} = +1$ (recollapsing) stiff-fluid Kantowski-Sachs model is given by [71]

$$A(t) = b, \tag{2.3.21}$$

$$B(t) = \frac{\sqrt{M}}{b}\sqrt{1 - \frac{b^2}{M}(t-t_0)^2}, \tag{2.3.22}$$

with (b, M, t_0) constants. The constant M appears in the density of stiff-fluid matter conservation law

$$p = \rho = \frac{M}{A^2 B^4}, \tag{2.3.23}$$

where p is the pressure and ϱ is the energy density. The Kantowski-Sachs model described by scale factors (2.3.21)–(2.3.22) begins and ends at "barrel" singularities ($A =$ const., $B = 0$) [53]. For simplicity we will take $t_0 = \sqrt{M}/b$

from now on, so that the range of t is $t \in [0, 2\sqrt{M}/b]$. For the exact solution (2.3.21)–(2.3.22), equations (2.3.9) and (2.3.12) can be integrated to give

$$t(\tau) = \frac{\sqrt{M}}{b} + \frac{\sqrt{k^2+M}}{b}\,\sin\left(\tau - \arcsin\sqrt{\frac{M}{k^2+M}}\right), \qquad (2.3.24)$$

$$r(\tau) = \frac{k}{b^2}\tau + r_0, \qquad (2.3.25)$$

where we took boundary conditions such that $t(0) = 0$. Having this, one can express B in terms of τ, i.e.,

$$B(\tau) = \frac{\sqrt{M}}{b}\sqrt{1 - \frac{k^2+M}{M}\sin^2\left(\tau - \arcsin\sqrt{\frac{M}{k^2+M}}\right)}, \qquad (2.3.26)$$

which in the limit $k = 0$ ($r =$ const.) gives

$$B(\tau) = \frac{\sqrt{M}}{b}\sin\tau. \qquad (2.3.27)$$

According to (2.3.10), the invariant string size is simply $S(\tau) = 2\pi B(\tau)$, which means that the string trivially starts with zero size, then expands to a maximum size and finally contracts to zero size again together with the universe, i.e. at $\tau = 2\arcsin\sqrt{M/(k^2+M)}$. This again comes from the fact that the string simply winds around the equatorial plane of the 2-sphere.

B. Null strings

The above solutions for tensile strings were all obtained for $\theta = \pi/2$. However, for the null strings, $\lambda = 0$, we can easily integrate (2.3.7), still keeping the general form of $\theta = \theta(\tau)$ in the ansatz (2.3.4), to obtain [78]

$$\dot{\theta} = \frac{d\theta}{d\tau} = \frac{l}{B^2}, \qquad (2.3.28)$$

with $l =$ const. The Eq. (2.3.8) now becomes

$$\dot{t}^2 = \frac{k^2}{A^2} + \frac{l^2}{B^2}, \qquad (2.3.29)$$

or, explicitly

$$\tau(t) = \int^t \frac{|AB|\ dt}{\sqrt{l^2A^2 + k^2B^2}}, \tag{2.3.30}$$

while (2.3.9) becomes

$$r(t) = k\int^t \frac{|B|\ dt}{|A|\sqrt{l^2A^2 + k^2B^2}}. \tag{2.3.31}$$

a) Λ-term solutions

Inserting (2.3.14)–(2.3.15) into (2.3.31), we have an exact relation between the spacetime and the string time coordinates

$$H_0 t(\tau) = \text{arccosh}\sqrt{\frac{l^2 + [H_0^2(k^2+l^2)\tau + |k|]^2}{k^2+l^2}}, \tag{2.3.32}$$

where we choose again boundary conditions such that $t(0) = 0$. Having given (2.3.32), one can write down (2.3.14)–(2.3.15) in terms of the τ-coordinate as

$$A(\tau) = \frac{1}{H_0}\sqrt{\frac{-k^2 + [H_0^2(k^2+l^2)\tau + |k|]^2}{k^2+l^2}}, \tag{2.3.33}$$

$$B(\tau) = \frac{1}{H_0}\sqrt{\frac{l^2 + [H_0^2(k^2+l^2)\tau + |k|]^2}{k^2+l^2}}. \tag{2.3.34}$$

This allows us to integrate (2.3.9) and (2.3.28). We obtain

$$r(\tau) = -\frac{k}{|k|}\text{arccth}\left(1 + \frac{H_0^2}{|k|}(k^2+l^2)\tau\right) + r_0, \tag{2.3.35}$$

$$\theta(\tau) = -\frac{l}{|l|}\text{arctg}\left(\frac{|k|}{|l|} + \frac{H_0^2}{|l|}(k^2+l^2)\tau\right) + \theta_0, \tag{2.3.36}$$

with $r_0, \theta_0 = \text{const}$. The invariant string size (2.3.10) is given by

$$\begin{aligned} S(\tau) &= 2\pi|B(\tau)\sin\theta(\tau)| \\ &= \frac{2\pi}{H_0\sqrt{k^2+l^2}}\left|\left(|k| + H_0^2(k^2+l^2)\tau\right)\sin\theta_0 - l\cos\theta_0\right|. \end{aligned} \tag{2.3.37}$$

It is useful to consider some special cases. For $l = 0$ (i.e., $\theta = \theta_0$), we have

$$A(\tau) = \sqrt{H_0^2 k^2 \tau^2 + 2|k|\tau}, \tag{2.3.38}$$

$$B(\tau) = H_0|k|\tau + H_0^{-1}, \tag{2.3.39}$$

$$t(\tau) = \frac{1}{H_0}\text{arccosh}(H_0^2|k|\tau + 1), \tag{2.3.40}$$

$$r(\tau) = -\frac{k}{|k|}\text{arccth}(H_0^2|k|\tau + 1) + r_0, \tag{2.3.41}$$

$$S(\tau) = \frac{2\pi}{H_0}|\sin\theta_0|(H_0^2|k|\tau + 1). \tag{2.3.42}$$

That is, the string winds around the 2-sphere at the angle $\theta = \theta_0$ and expands to infinite size together with the scale factor.

Another special case is given in the limit $k = 0$ (i.e., $r = r_0$), where we have

$$A(\tau) = H_0|l|\tau, \tag{2.3.43}$$

$$B(\tau) = \frac{1}{H_0}\sqrt{1 + H_0^4 l^2 \tau^2}, \tag{2.3.44}$$

$$t(\tau) = \frac{1}{H_0}\text{arccosh}\sqrt{1 + H_0^4 l^2 \tau^2}, \tag{2.3.45}$$

$$\theta(\tau) = -\frac{l}{|l|}\text{arctg}(H_0^2|l|\tau) + \theta_0, \tag{2.3.46}$$

$$S(\tau) = \frac{2\pi}{H_0}|H_0^2 l\tau\sin\theta_0 - \cos\theta_0|. \tag{2.3.47}$$

Notice that when τ varies from 0 to ∞, the angle θ changes by $\pi/2$. Therefore we can distinguish a number of scenarios: 1) $\theta_0 = 0$. In this case, the string starts at the equatorial plane and then moves towards one of the poles in such a way that its size $S(\tau)$ is constant, i.e., the contraction of the string is exactly balanced by the expansion of the 2-sphere. 2) $\theta_0 \neq 0$. The string starts somewhere on one hemisphere, then crosses the equator and approaches a fixed position on the other hemisphere. During its evolution, the string grows indefinitely. 3) $\theta_0 \neq 0$. The string starts somewhere on one hemisphere, then moves towards the nearest pole where it collapses. It then reappears and approaches a fixed position on the same hemisphere. After the collapse, and during its later evolution, its size will grow indefinitely.

Returning to the general expression (2.3.37), it is easy to see that the dynamics in the general case is qualitatively similar to the $k=0$ case, although quantitatively different. For instance, if $\theta_0 = 0$, the string starts at a fixed angle $\theta(\tau=0) = -\text{sign}(l)\text{arctg}(|k|/|l|)$, and can then approach the nearest pole in such a way that its size is constant. Similarly, we can also find solutions where the string expands indefinitely, possibly after collapsing once during its early evolution.

b) Time-symmetric stiff-fluid solutions

Using (2.3.21)–(2.3.22) for the null string case, we obtain from (2.3.30)

$$\tau(t) = \int_0^t \sqrt{\frac{M - b^2(t-\sqrt{M}/b)^2}{l^2b^2 + k^2M(1-b^2(t-\sqrt{M}/b)^2)/b^2}}\, dt, \tag{2.3.48}$$

where we took again boundary conditions such that $t(0) = 0$. From (2.3.9), we get

$$r(\tau) = \frac{k}{b^2}\tau + r_0. \tag{2.3.49}$$

Thus r is expressed directly as a function of τ, while (2.3.48) must be inverted to give $t(\tau)$. Notice that (2.3.48) is an elliptic integral, which can be given in terms of elliptic functions [1]

$$\begin{aligned}\tau(t) &= \frac{b}{|k|}\sqrt{\frac{M}{b^2}+\frac{l^2b^2}{k^2}}\; E\left(\xi, \sqrt{\frac{Mk^2}{Mk^2+l^2b^4}}\right) \\ &- \frac{l^2b^4}{k^2\sqrt{Mk^2+l^2b^4}} F\left(\xi, \sqrt{\frac{Mk^2}{Mk^2+l^2b^4}}\right) \\ &- \frac{b}{|k|}\left(\frac{\sqrt{M}}{b} - t\right)\sqrt{\frac{M/b^2-(t-\sqrt{M}/b)^2}{M/b^2+l^2b^2/k^2-(t-\sqrt{M}/b)^2}},\end{aligned} \tag{2.3.50}$$

where F, E are the elliptic integrals of first and second kind, respectively, and

$$\xi = \arcsin\left[\sqrt{\frac{Mk^2 + l^2b^4}{Mk^2}}\sqrt{\frac{M/b^2 - (t-\sqrt{M}/b)^2}{M/b^2 + l^2b^2/k^2 - (t-\sqrt{M}/b)^2}}\right]. \quad (2.3.51)$$

Here we analyze some special cases in which (2.3.50) becomes elementary. First, $l = 0$, gives the simple solution

$$t = \frac{|k|}{b}\tau, \quad (2.3.52)$$

$$r = \frac{k}{b^2}\tau + r_0, \quad (2.3.53)$$

$$\theta = \theta_0. \quad (2.3.54)$$

Then the string size is

$$S(\tau) = \frac{2\pi}{b}|\sin\theta_0|\sqrt{2|k|\sqrt{M}\tau - k^2\tau^2}, \quad (2.3.55)$$

that is, the string starts with zero size, then expands and eventually collapses together with the universe.

Another special case is when $k = 0$. Then,

$$\tau(t) = \frac{M}{2|l|b^2} \times$$

$$\times\left[\left(\frac{bt}{\sqrt{M}} - 1\right)\sqrt{1 - \left(\frac{bt}{\sqrt{M}} - 1\right)^2} + \arcsin\left(\frac{bt}{\sqrt{M}} - 1\right) + \frac{\pi}{2}\right], \quad (2.3.56)$$

$$r(\tau) = r_0 = \text{const.}, \quad (2.3.57)$$

$$\theta(t) = \frac{l}{|l|b}\arcsin\left(\frac{bt}{\sqrt{M}} - 1\right) + \theta_0. \quad (2.3.58)$$

In this case the string size is given by

$$S(t) = \frac{2\pi\sqrt{M}}{b}\sqrt{1 - \left(\frac{bt}{\sqrt{M}} - 1\right)^2} \times$$

$$\times \left|\sqrt{1 - \left(\frac{bt}{\sqrt{M}} - 1\right)^2}\sin\theta_0 \pm \left(\frac{bt}{\sqrt{M}} - 1\right)\cos\theta_0\right|. \quad (2.3.59)$$

If $\theta_0 = 0$, the string is at the equator and it simply follows the evolution of the universe, i.e., it starts with zero size, then expands and eventually collapses together with the universe. On the other hand, if $\theta_0 \neq 0$, the string has the possibility to pass one of the poles of the 2-sphere, i.e. it starts with zero size, then expands but collapses, then expands again and eventually collapses together with the universe.

It is straightforward to check that the qualitative behaviour of the string solutions in the general case (described by (2.3.50)) essentially follows the $k = 0$ case, thus we shall not go into the quantitative details here.

2.3.2 Strings in Bianchi I spacetime

The validity of the superstring effective equations for strings in Bianchi type homogeneous spacetimes will be discussed in Chapter 3, and in this subsection we consider the evolution of strings based on the equations (2.1.18)–(2.1.20) in Bianchi I background spacetimes with the metric [186]

$$ds^2 = dt^2 - X^2(t)dx^2 - Y^2(t)dy^2 - Z^2(t)dz^2, \tag{2.3.60}$$

where X, Y, Z are the scale factors. Comparing with equations (2.3.1)–(2.3.4) in the case $\theta = \pi/2$, a natural first attempt of an ansatz is now [78]

$$X^0 = t(\tau), \qquad X^1 = x = f(\tau)\cos\sigma, \qquad X^2 = y = g(\tau)\sin\sigma, \qquad X^3 = z = \text{const.}, \tag{2.3.61}$$

and the invariant string size is

$$S(\tau) = \int_0^{2\pi} \sqrt{f^2(\tau)X^2(t(\tau))\sin^2\sigma + g^2(\tau)Y^2(t(\tau))\cos^2\sigma}\, d\sigma. \tag{2.3.62}$$

The ansatz (2.3.61) describes a closed string of "elliptic-shape", in the sense that

$$\frac{x^2}{f^2} + \frac{y^2}{g^2} = 1, \tag{2.3.63}$$

i.e., it generalizes the circular string ansatz considered earlier. This seems to be the most natural ansatz in the spacetimes with the line element (2.3.60) because of the nonzero shear tensor [118]. The equations and constraints read as

$$\ddot{t}+XX_{,t}\left(\dot{f}^2\cos^2\sigma-\lambda f^2\sin^2\sigma\right)+YY_{,t}\left(\dot{g}^2\sin^2\sigma-\lambda g^2\cos^2\sigma\right)=0, \quad (2.3.64)$$

$$\ddot{f}+\lambda f+2\frac{X_{,t}}{X}\dot{t}\dot{f} = 0, \quad (2.3.65)$$

$$\ddot{g}+\lambda g+2\frac{Y_{,t}}{Y}\dot{t}\dot{g} = 0, \quad (2.3.66)$$

as well as

$$X^2\dot{f}f-Y^2\dot{g}g=0, \quad (2.3.67)$$

$$\dot{t}^2-X^2\left(\dot{f}^2\cos^2\sigma+\lambda f^2\sin^2\sigma\right)-Y^2\left(\dot{g}^2\sin^2\sigma+\lambda g^2\cos^2\sigma\right)=0. \quad (2.3.68)$$

Notice, however, that the equations (2.3.64–(2.3.68) are not all independent. After some algebra, one finds that they reduce to just four independent equations

$$\begin{aligned}\ddot{f}+\lambda f+2\frac{X_t}{X}\dot{t}\dot{f}&=0,\\ \dot{t}^2&=X^2\dot{f}^2+\lambda Y^2g^2,\\ X^2f\dot{f}&=Y^2g\dot{g},\\ X^2\dot{f}^2+\lambda Y^2g^2&=Y^2\dot{g}^2+\lambda X^2f^2.\end{aligned} \quad (2.3.69)$$

The last two equations of (2.3.69) lead to the following two possibilities:

a:

$$\frac{\dot{f}}{f}=\frac{\dot{g}}{g} \quad \text{and} \quad X^2f\dot{f}=Y^2g\dot{g}, \quad (2.3.70)$$

which are solved by

$$X(t)=\pm c_1Y(t), \quad g(\tau)=c_1f(\tau), \quad (2.3.71)$$

where c_1 is a constant. After a trivial coordinate redefinition, this corresponds to a circular string in an axially symmetric background.

b:
$$\frac{\dot{f}}{f} = -\lambda\frac{g}{\dot{g}} \quad \text{and} \quad X^2 f\dot{f} = Y^2 g\dot{g}, \tag{2.3.72}$$
from which follows that
$$-\lambda X^2 f^2 = Y^2 \dot{g}^2. \tag{2.3.73}$$
This equation has no real solutions for tensile strings ($\lambda = 1$), while for null strings ($\lambda = 0$), we find
$$f = \text{const.} \equiv c_1, \quad g = \text{const.} \equiv c_2, \quad t = \text{const.} \equiv c_3, \tag{2.3.74}$$
with $X(t)$, $Y(t)$ arbitrary. Such solutions, with $t =$ const., have been considered before in other contexts [193], but since they do not fulfil the physical requirement of forward propagation ($\dot{t} > 0$), we discard them here.

Thus our ansatz (2.3.61) eventually only works in the case (2.3.71). Then the equations (2.3.69) reduce to
$$\ddot{f} + \lambda f + 2\frac{X_t}{X}\dot{t}\dot{f} = 0, \tag{2.3.75}$$
$$\dot{t}^2 = X^2\dot{f}^2 + \lambda X^2 f^2. \tag{2.3.76}$$
For the null strings ($\lambda = 0$), they are immediately solved by
$$\tau = \frac{1}{|c_1|}\int_0^t X(t)dt, \tag{2.3.77}$$
$$f(t) = c_1\int_{t_0}^t \frac{dt}{X^2(t)}. \tag{2.3.78}$$
For the tensile strings ($\lambda = 1$), they can not be solved in general. However, the same equations appeared in a study of strings in Friedmann-Robertson-Walker universes [193], and some special solutions were found there.

Here we are interested in strings in Bianchi universes. Usually, one starts with the Kasner-type vacuum power-law solutions [142], which are given by
$$X(t) = t^{p_1}, \tag{2.3.79}$$
$$Y(t) = t^{p_2}, \tag{2.3.80}$$
$$Z(t) = t^{p_3}, \tag{2.3.81}$$

and

$$p_1 + p_2 + p_3 = 1 \qquad\qquad p_1^2 + p_2^2 + p_3^2 = 1, \tag{2.3.82}$$

where

$$-\frac{1}{3} \le p_1 \le 0, \qquad 0 \le p_2 \le \frac{2}{3}, \qquad \frac{2}{3} \le p_3 \le 1. \tag{2.3.83}$$

However, as we saw before, our string ansatz only works in axially symmetric cases. Furthermore, we shall usually also allow the presence of matter.

A special case of the model (2.3.60) is an axially symmetric Kasner model in which the matter is that of the stiff-fluid. The metric reads

$$ds^2 = dt^2 - A^2(t)\left(dx^2 + dy^2\right) - Z^2(t)dz^2, \tag{2.3.84}$$

and it is just Kantowski-Sachs metric (2.3.1) of zero curvature. For the stiff-fluid, $p = \rho$, the conservation law is given by $\rho Z^2 A^4 = \rho t^2 = k^2/16\pi$, which gives the solutions for the scale factors in the form [191]

$$A(t) = t^{p_A}, \tag{2.3.85}$$
$$Z(t) = t^{1-2p_A}, \tag{2.3.86}$$

where

$$p_A = \frac{1}{3}[1 \pm \sqrt{1 - 3k^2/2}\,], \tag{2.3.87}$$

that is, $0 \le p_A \le 2/3$.

Under the ansatz (2.3.61) with $f = g$, the equations of motion (2.3.75)–(2.3.76) become

$$\ddot{f} + \lambda f + \frac{2p_A}{t}\dot{t}\dot{f} = 0, \tag{2.3.88}$$
$$\dot{t}^2 - t^{2p_A}(\dot{f}^2 + \lambda f^2) = 0. \tag{2.3.89}$$

A. Tensile strings

For the tensile strings ($\lambda = 1$), equations (2.3.88)–(2.3.89) were considered in [193]. They do not seem to be integrable, but some special solutions were found

$$t(\tau) = A\exp(c_1\tau), \quad f(\tau) = B\exp(c_2\tau), \tag{2.3.90}$$

where the constants (A, B, c_1, c_2) are given by

$$c_1 = \frac{\mp 1}{\sqrt{(p_A - 1)(p_A + 1)}}, \quad c_2 = \pm\sqrt{\frac{p_A - 1}{p_A + 1}}, \quad B = \frac{A^{1-p_A}}{\sqrt{2p_A(p_A - 1)}}. \tag{2.3.91}$$

However, this solution is not real for the values allowed in our case $(0 \leq p_A \leq 2/3)$, so it must be discarded.

All we can do then is to determine the asymptotics of the solutions to equations (2.3.88)–(2.3.89). One finds for $\tau \to \infty$

$$t(\tau) = A\tau, \tag{2.3.92}$$

$$f(\tau) = A^{1-p_A}\tau^{-p_A}\cos\tau. \tag{2.3.93}$$

The invariant string size reads as $(\tau \to \infty)$

$$S(\tau) = 2\pi A|\cos\tau|, \tag{2.3.94}$$

so it asymptotically oscillates with constant amplitude and unit frequency, while the comoving string size goes to zero.

B. Null strings

For the null strings $(\lambda = 0)$ in axially symmetric Kasner spacetime, the equations (2.3.77)–(2.3.78) are integrated to give

$$t(\tau) = [|c_1|(1 + p_A)\tau]^{\frac{1}{1+p_A}}, \tag{2.3.95}$$

$$f(\tau) = c_1\left(\frac{1 + p_A}{1 - p_A}\right)[|c_1|(1 + p_A)\tau]^{\frac{1-p_A}{1+p_A}}. \tag{2.3.96}$$

In this case the invariant string size is

$$S(\tau) = 2\pi|c_1|\left(\frac{1 + p_A}{1 - p_A}\right)[|c_1|(1 + p_A)\tau]^{\frac{1}{1+p_A}}, \tag{2.3.97}$$

which blows up for $\tau \to \infty$. This is also the case for the comoving string size.

We close this subsection with some comments on the possibility of having elliptic-shaped strings in anisotropic Bianchi backgrounds. As we saw, the ansatz (2.3.61) led to inconsistencies unless $f(\tau) = g(\tau)$ and $X(t) = Y(t)$. However, this does not mean that we must completely rule out the possibility of having elliptic-shaped string configurations. In fact, it is possible to make an ansatz more general than (2.3.61), but still describing an elliptic-shaped string. This can be done along the lines of the procedure used in reference [147] (in a somewhat different context): We discard the conformal gauge and work directly with the Nambu-Goto action. In that case, the ansatz (2.3.61) leaves us more freedom than before. Unfortunately, the equations of motion now become more complicated than before, but at least they are not explicitly inconsistent, and there is some hope that one can find special solutions or at least solve the equations numerically.

This more general ansatz consists in replacing (2.3.61) by

$$\begin{aligned} X^0 &= t(\tau), \quad X^1 = x = f(\tau)\cos\phi(\tau,\sigma), \\ X^2 &= y = g(\tau)\sin\phi(\tau,\sigma), \quad X^3 = z = \text{const.}, \end{aligned} \tag{2.3.98}$$

and the function $\phi(\tau, \sigma)$ gives us the extra freedom as mentioned above.

2.3.3 Strings in axisymmetric Bianchi IX spacetime

Another interesting example of a curved background for strings to be considered is the Bianchi type IX (BIX) background. It generalizes the $\mathcal{K} = +1$ isotropic Friedmann model to the case of anisotropic spacetimes. It has been shown that all BIX models recollapse similarly as $\mathcal{K} = +1$ Friedmann models [153, 154]. The general case cannot be solved analytically for the scale factors and they subject to chaotic behaviour. Because of that we consider only an axially symmetric Bianchi IX model. The metric of such a model (in a holonomic/coordinate frame), is given by [10, 14, 35] (see also Appendix A)

$$ds^2 = dt^2 - c^2(t)\,(d\psi + \cos\theta d\varphi)^2 - a^2(t)\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{2.3.99}$$

where ψ, θ, φ are the Euler angles ($0 \le \psi \le 4\pi, 0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$). We use the following ansatz for the spacetime coordinates

$$X^0 = t(\tau), \quad X^1 = \psi(\tau), \quad X^2 = \theta(\tau), \quad X^3 = \varphi = \sigma, \tag{2.3.100}$$

and the equations of motion (2.1.18) then read

$$\ddot{t} + cc_{,t}\dot{\psi}^2 + aa_{,t}\dot{\theta}^2 - \lambda\left(cc_{,t}\cos^2\theta + aa_{,t}\sin^2\theta\right) = 0, \quad (2.3.101)$$

$$\ddot{\psi} + 2\frac{c_{,t}}{c}\dot{t}\dot{\psi} + \frac{c^2}{a^2}\cot\theta\dot{\psi}\dot{\theta} = 0, \quad (2.3.102)$$

$$\ddot{\theta} + 2\frac{a_{,t}}{a}\dot{t}\dot{\theta} - \lambda\frac{c^2 - a^2}{a^2}\sin\theta\cos\theta = 0, \quad (2.3.103)$$

$$\frac{c^2}{a^2}\frac{1}{\sin\theta}\dot{\psi}\dot{\theta} = 0. \quad (2.3.104)$$

The last of these equations (2.3.104) says that either $\psi =$ const. or $\theta =$ const. $(\theta \neq 0)$. The constraints (2.1.19)–(2.1.20) read

$$\dot{t}^2 - c^2\dot{\psi}^2 - a^2\dot{\theta}^2 - \lambda\left(a^2\sin^2\theta + c^2\cos^2\theta\right) = 0, \quad (2.3.105)$$

$$c^2\cos\theta\dot{\psi} = 0, \quad (2.3.106)$$

from which it follows that either ψ must be constant or $\theta = \pi/2$. We will consider both cases. For each of these cases the invariant string size is given by

$$S(\tau) = 2\pi\sqrt{a^2(t(\tau))\sin^2\theta(\tau) + c^2(t(\tau))\cos^2\theta(\tau)}. \quad (2.3.107)$$

The well-known stiff-fluid solutions of the Einstein equations for Bianchi IX axisymmetric model are given by [10, 14]

$$c^2 = \frac{A}{\cosh A\eta}, \quad (2.3.108)$$

$$a^2 = \frac{B^2\cosh A\eta}{4A\cosh^2(\frac{B}{2}\eta)}, \quad (2.3.109)$$

$$p = \rho = \frac{M^2}{4a^4c^2}, \quad (2.3.110)$$

where A, B, M are constants $(A > |B|)$ with

$$B^2 = A^2 - M^2, \quad (2.3.111)$$

$$t = \int^{\eta} a^2|c|d\eta. \quad (2.3.112)$$

One can easily see that the vacuum solution $M = 0$ is given by

$$c^2 = \frac{A}{\cosh A\eta}, \tag{2.3.113}$$

$$a^2 = \frac{A\cosh A\eta}{4\cosh^2(\frac{A}{2}\eta)}. \tag{2.3.114}$$

Notice that the scale factor $c(\eta)$ increases from $c(-\infty) = 0$ to $c(0) = c_{\text{max}} = \sqrt{A}$ and then decreases to $c(\infty) = 0$ again. On the other hand, the scale factor $a(\eta)$ decreases from $a(-\infty) = \sqrt{A/2}$ to $a(0) = a_{\text{min}} = \sqrt{A/4}$ and then increases to $a(\infty) = \sqrt{A/2}$ again. However, the volume essentially follows the scale factor $c(\eta)$, i.e., the universe is of recollapsing type.

A. $\theta = \pi/2$ solutions

If we assume $\theta = \pi/2$ and $\dot{\psi} \neq 0$, then we can easily integrate (2.3.102) to give

$$\dot{\psi} = \frac{m}{c^2}, \tag{2.3.115}$$

with $m =$ const. Then we have

$$\tau(t) = \int_0^t \frac{|c|dt}{\sqrt{m^2 + \lambda a^2 c^2}}, \tag{2.3.116}$$

$$\psi(t) = m\int_{t_0}^t \frac{dt}{|c|\sqrt{m^2 + \lambda a^2 c^2}}. \tag{2.3.117}$$

For the null strings ($\lambda = 0$), we can easily get the solution for the vaccum case $M = 0$ in terms of parametric time η, by using (2.3.113) and (2.3.114), i.e.,

$$d\eta = \frac{|m|}{a^2 c^2} d\tau, \tag{2.3.118}$$

which is integrated to

$$\eta(\tau) = \frac{2}{A}\text{arcth}\left(\frac{2|m|}{A}\tau\right), \tag{2.3.119}$$

where we took boundary conditions such that $\eta(0) = 0$. The solutions for the scale factors $c(\tau)$ and $a(\tau)$ are given by

$$c^2(\tau) = A\left(\frac{A^2 - 4m^2\tau^2}{A^2 + 4m^2\tau^2}\right), \tag{2.3.120}$$

$$a^2(\tau) = \frac{1}{4A}\left(A^2 + 4m^2\tau^2\right), \tag{2.3.121}$$

and $\tau^2 \leq A^2/4m^2$. Then we find

$$t(\tau) = \frac{|m|}{\sqrt{A}}\int_0^\tau d\tau\sqrt{\frac{A^2 + 4m^2\tau^2}{A^2 - 4m^2\tau^2}}, \tag{2.3.122}$$

$$\psi(\tau) = \psi_0 + \frac{m}{A}\int_0^\tau d\tau\left(\frac{A^2 + 4m^2\tau^2}{A^2 - 4m^2\tau^2}\right). \tag{2.3.123}$$

The integral for $t(\tau)$ is of elliptic type, while the one for $\psi(\tau)$ is elementary. However, we shall not need the explicit results here.

The invariant string size reads

$$S(\tau) = \frac{\pi}{\sqrt{A}}\sqrt{A^2 + 4m^2\tau^2}. \tag{2.3.124}$$

From the above, we conclude that for the admissible values of the parameter τ, the string starts with the size $S = \pi\sqrt{2A}$ for $\tau = -A/2|m|$, then contracts to the size $S = \pi\sqrt{A}$ for $\tau = 0$, and expands again to $S = \pi\sqrt{2A}$ for $\tau = A/2|m|$. This can be easily understood physically since, for $\theta = \pi/2$, the string is winding in the φ-direction with scale factor a.

B. $\dot{\psi} = 0, \dot{\theta} \neq 0$ solutions

The equations (2.3.105)–(2.3.106) become

$$\ddot{t} + aa_{,t}\dot{\theta}^2 - \lambda\left(cc_{,t}\cos^2\theta + aa_{,t}\sin^2\theta\right) = 0, \tag{2.3.125}$$

$$\ddot{\theta} + 2\frac{a_{,t}}{a}\dot{t}\dot{\theta} - \lambda\frac{c^2 - a^2}{a^2}\sin\theta\cos\theta = 0, \tag{2.3.126}$$

$$\dot{t}^2 - a^2\dot{\theta}^2 - \lambda\left(a^2\sin^2\theta + c^2\cos^2\theta\right) = 0. \tag{2.3.127}$$

Notice that the first equation can be obtained from the two others. Thus we have just two coupled equations; one of first order and one of second order. For the tensile strings, the general solution is not known. For the null strings, we can integrate for $\theta(\tau)$ and $t(\tau)$

$$\dot{\theta} = \frac{s}{a^2}, \tag{2.3.128}$$

$$\dot{t}^2 = \frac{s^2}{a^2}, \tag{2.3.129}$$

with $s =$ const. Then using the exact vacuum solutions (2.3.113)–(2.3.114), we can integrate this further since $(d\eta/d\tau)^2 = s^2/a^6c^2$. One finds

$$|s|d\tau = \frac{A^2}{8}\frac{\cosh A\eta}{\cosh^3\left(\frac{A}{2}\eta\right)}d\eta, \tag{2.3.130}$$

which can be integrated explicitly in terms of elementary functions. In principle we can then also obtain expressions for $t(\tau)$ and $\theta(\tau)$. However, it turns out to be somewhat simpler to express everything in terms of the conformal time η. For instance

$$d\theta = \frac{s}{|s|}|ac|d\eta, \tag{2.3.131}$$

which leads to

$$\theta(\eta) - \theta_0 = 2\frac{s}{|s|}\text{arctg}\left(e^{A\eta/2}\right). \tag{2.3.132}$$

It is then straightforward to write down an explicit expression for the invariant string size similar to equation (2.3.107), but with S as a function of η, since the scale factors are already given in terms of η.

It follows from the above results that during the whole evolution of the universe, the polar angle θ changes by π. Thus there are two scenarios: If $\theta_0 = 0$, then the string starts with zero size at one of the poles, then expands and eventually collapses to zero size again at the other pole. On the other hand, if $\theta_0 \neq 0$, then the string starts with finite size, passes one of the poles (still with finite size) and eventually ends up with a finite size. Thus the behaviour is qualitatively similar to that of strings in Kantowski-Sachs spacetimes as described in subsection 2.3.1.

Our results demonstrate the richness of evolution schemes for extended objects, here strings, in curved backgrounds. For the tensile strings, this is due to the "competition" between the string tension and the gravitational field, which together determines the evolution of the string. For the null strings, it is simply due to the fact that we are dealing with an extended object in a gravitational field, i.e., the object subjects to tidal forces. In both cases, the situation should be compared with the conceptually much simpler problems of point particle propagation in curved spacetimes and string evolution and propagation in flat Minkowski spacetime.

We mainly considered closed circular strings, which allowed us to obtain simple exact analytical results in most cases. We essentially saw three qualitatively different kinds of circular string evolution: a) the string simply follows the expansion or contraction of the universe, b) the string makes a finite or infinite number of oscillations during the evolution of the universe, c) the contraction of the string is exactly balanced by the expansion of the universe, such that the physical string size is constant.

We also discussed the problems of obtaining consistent equations of motion describing an elliptic-shaped string configuration. The use of the simplest, and a priori most natural ansatz, describing an elliptic-shaped string, led to inconsistent equations of motion, but the application of the more complicated ansatz, as described at the end of subsection 2.3.2, with its more complicated equations of motion, perhaps could solve these problems.

Chapter 3

Superstring cosmologies

3.1 Superstring effective actions

As mentioned in Chapter 1, there is bosonic string theory and essentially five different superstring theories (one for open and other for closed strings[1]) plus supergravity (in eleven dimensions) as a low-energy limit of M-theory [110, 152, 178]. The latter will be the topic of our Chapter 5, while the former we discuss in this Chapter 3.

From the action (2.1.1) one can learn that the bosonic degrees of freedom fulfil a 2-dimensional wave equation (cf. (2.1.18) for $\Gamma^{\mu}_{\nu\rho} = 0$) and fermionic degrees of freedom fulfil a 2-dimensional Dirac equation

$$\gamma^{a}\partial_{a}\psi^{\mu} = 0. \tag{3.1.1}$$

In both bosonic and fermionic case, one is able to distinguish between left and right moving modes (for closed strings) and they can be treated separately. In the case of fermions one separates into the modes which have different chirality – ψ^{μ}_{L} has negative chirality and ψ^{μ}_{R} has positive chirality. Open string theory has only one sector – due to the reflection at free ends, left and right-moving modes are the same.

As for closed strings one may impose boundary conditions which can be

[1] In fact, open string theory is contained in closed string theory once one turns the interaction beetween free ends which can make the open strings become closed [177, 178].

either periodic or antiperiodic. For fermions, for instance, they read

$$\psi^{\mu}_{L,R}(\sigma,\tau) = \pm\ \psi^{\mu}_{L,R}(\sigma + 2\pi, \tau). \tag{3.1.2}$$

The periodic boundary conditions sector is called Ramond (R) sector and the antiperiodic boundary conditions sector is called Neuveu-Schwarz (NS) sector. The combinations of these give four possible sectors in total: NS-NS and R-R sectors give bosonic excitations while NS-R and R-NS give fermionic excitations. These excitations can be either massless or massive and in the limit of $\alpha' \to 0$ one is able to construct effective theories based entirely on massless or very light degrees of freedom while massive modes are decoupled (or integrated out).

The formal procedure in order to obtain an effective action for the massless excitations, requires the application of the world-sheet action (2.1.1) for curved background as in (2.1.3) (a generalized non-linear σ-model). By imposing the constraint that quantum corrections do not break conformal (Weyl) invariance [42, 178] (in other words, that the 2-dimensional world-sheet stress-energy tensor is traceless) one gets the renormalization β-functions equations to vanish. These β-function equations, in turn, can be derived from an effective action which, although not determined uniquely, can always give a unique set of field equations upon variation. All superstring theories are consistent in $D = 10$ spacetime dimensions.

The main feature of all superstring effective actions is the appearance of the non-minimal coupling of massless modes to the dilaton ϕ (which is also a 'stringy' mode) which is involved into the string-loop expansion parameter $g_s = \exp(\phi/2)$ (cf. (1.1.3)). One should emphasize that in string theory there are two 'orthogonal' perturbation expansions [110]. These are: the string-loop expansion which is the true quantum mechanical expansion and at each order of the string-loop expansion there is a 2-dimensional field theory evaluated in loop expansion connected with the fundamental string length $\lambda_s = \sqrt{\alpha'}$ (1.1.2). In this book we do not consider string loop expansion restricting ourselves only to lowest orders expansion in α'.

Let us start with bosonic *type IIA* superstring effective action. It has $N = 2$

supersymmetries of opposite chirality and reads [152][2]

$$S_{\rm IIA} = \frac{1}{2\lambda_s^8}\left\{\int d^{10}x\sqrt{-g_{10}}\left[e^{-\phi}\left(R_{10} + (\nabla\phi)^2 - \frac{1}{12}H_3^2\right)\right.\right.$$
$$\left.\left. - \frac{1}{4}F_2^2 - \frac{1}{48}(F_4{}')^2\right] + \frac{1}{2}\int B_2 \wedge F_4 \wedge F_4\right\} \qquad (3.1.3)$$

where R_{10} is the Ricci scalar curvature of the spacetime with metric $g_{\mu\nu}$ and $g_{10} \equiv \det g_{\mu\nu}$. Strings sweep out geodesic surfaces with respect to the metric $g_{\mu\nu}$. The antisymmetric tensor field strengths are defined by $H_3 = dB_2$, $F_2 = dA_1$, $F_4 = dA_3$ and $F_4' = F_4 + A_1 \wedge H_3$, where A_1, B_2, A_3 denote antisymmetric 1-, 2- and 3-form potentials and d is the exterior derivative. The last term in (3.1.3) is a Chern-Simons term and is a necessary consequence of supersymmetry [178]. As we mentioned already the action (3.1.3) represents the zeroth-order expansion in both the string coupling g_s and the inverse string tension α'. The NS-NS sector of the action contains the graviton, $g_{\mu\nu}$, the antisymmetric 2-form potential, B_2, and the dilaton field, ϕ. The RR sector contains antisymmetric 1- and 3-form potentials. The NS-NS sector couples directly to the dilaton, but the RR sector does not.

The bosonic *type IIB* superstring effective theory has $N = 2$ supersymmetries of the same chirality and its effective action reads

$$S_{\rm IIB} = \frac{1}{2\lambda_s^8}\left\{\int d^{10}x\sqrt{-g_{10}}\left[e^{-\phi}\left(R_{10} + (\nabla\phi)^2 - \frac{1}{12}(H_3^{(1)})^2\right)\right.\right.$$
$$\left.\left. - \frac{1}{2}(\nabla\chi)^2 - \frac{1}{12}(H_3^{(2)} + \chi H_3^{(1)})^2 - \frac{1}{240}(F_5)^2\right] + \int A_4 \wedge H_3^{(2)} \wedge H_3^{(1)}\right\} \qquad (3.1.4)$$

Here the bosonic massless excitations arising in the NS–NS sector are the dilaton, ϕ, the metric, $g_{\mu\nu}$, and the antisymmetric, 2-form potential, denoted by $B^{(1)}_{\mu\nu}$. The RR sector contains a scalar axion field, χ, a 2-form potential, $B^{(2)}_{\mu\nu}$ and the RR field strengths are defined by $H_3^{(2)} = dB_2^{(2)}$ and $F_5 = dA_4 + B_2^{(2)} \wedge H_3^{(1)}$.

In heterotic superstring theories supersymmetry is imposed only in the right-moving sector so these theories are $N = 1$ supersymmetric. Quantization

[2] In a D-dimensional spacetime one has $1/2\lambda_s^{D-2}$ in front of the action integral which reduces to the given result for $D = 10$.

of the left-moving sector requires the gauge groups to be either $SO(32)$ or $E_8 \times E_8$ and the choice of the group depends on the boundary conditions (3.1.2). The *heterotic* superstring effective action reads

$$S_{\rm H} = \frac{1}{2\lambda_s^8} \int d^{10}x\sqrt{-g_{10}}e^{-\phi}\left[R_{10} + (\nabla\phi)^2 - \frac{1}{12}H_3^2 - \frac{1}{4}F_2^2\right], \qquad (3.1.5)$$

where F_2^2 is the field strength corresponding to the gauge groups SO(32) or $\mathrm{E}_8 \times \mathrm{E}_8$ and $H_3 = dB_2$ is the field strength of a 2-form potential, B_2.

The last of the superstring theories is an open string theory or *type I* theory. Because of the obvious reason (left and right moving sectors must be the same) it has $N = 1$ supersymmetry and the effective action reads

$$S_{\rm I} = \frac{1}{2\lambda_s^8} \int d^{10}x\sqrt{-g_{10}}\left[e^{-\phi}\left(R_{10} + (\nabla\phi)^2\right) - \frac{1}{12}H_3^2 - \frac{1}{4}e^{-\phi/2}F_2^2\right], \quad (3.1.6)$$

where F_2^2 is the Yang-Mills field strength taking values in the gauge group $G = \mathrm{SO}(32)$ and $H_3 = dB_2$ is the field strength of a 2–form potential, B_2. We note that this field strength is not coupled to the dilaton field, and since both actions (3.1.5) and (3.1.6) have the same particle content this is the only difference between the two theories.

It is not difficult to notice that all the above theories (IIA, IIB, heterotic and I) as given by the formulas (3.1.3), (3.1.4), (3.1.5) and (3.1.6), have the common sector which (except different coupling of H_3 in type I) is

$$S = \frac{1}{2\lambda_s^8} \int d^{10}x\sqrt{-g_{10}}\; e^{-\phi}\left[R_{10} + (\nabla\phi)^2 - \frac{1}{12}H_3^2\right]. \qquad (3.1.7)$$

In the *bosonic* theory the action (3.1.7) is also valid except that the number of spacetime dimensions changes into $D = 26$, thus

$$S = \frac{1}{2\lambda_s^{24}} \int d^{26}x\sqrt{-g_{26}}\; e^{-\phi}\left[R_{26} + (\nabla\phi)^2 - \frac{1}{12}H_3^2\right]. \qquad (3.1.8)$$

The *common sector* (3.1.7)–(3.1.8), after a suitable dimensional reduction, (see e.g. [152]) will be the main point of the discussion of this Chapter. It is the sector which leads to the so-called pre-big-bang cosmological scenario based on the $D = 4$-dimensional effective action of the form

$$S = \frac{1}{2\lambda_s^2} \int d^4x \sqrt{-g_4}\ e^{-\phi} \left\{ R_4 + (\nabla\phi)^2 - \frac{1}{12} H^2 \right\}, \tag{3.1.9}$$

where ϕ is the dilaton, g_4 the determinant of the 4-dimensional metric, R_4 the 4-dimensional Ricci scalar, $H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]}$ ($dH = 0$) the NS-NS axion with $H^2 = H_{\mu\nu\rho} H^{\mu\nu\rho}$ and $B_{\mu\nu}$ the antisymmetric tensor potential, $B_{\mu\nu} = -B_{\nu\mu}$. The dilaton, the axion and the graviton are the massless ('stringy') modes of the theory.

The actions (3.1.3)–(3.1.9) are presented in the string frame which is characterised by the property that in this frame the fundamental string length $\lambda_s =$ const., while the Planck mass M_{pl} changes with time. The reduced action (3.1.9) without axion ($H = 0$) is, in fact, the same action as the action of Brans-Dicke theory in Jordan frame with a Brans-Dicke parameter $\omega = -1$ [34]. The Einstein limit is recoverable for constant dilaton. Introducing Brans-Dicke field Φ one can write down the sequence of relations

$$\Phi(t) = \frac{1}{G(t)} = \frac{e^{-\phi(t)}}{\lambda_s^2} = M_{pl}^2(t), \tag{3.1.10}$$

where t is time and G is Newton gravitational "constant".

In our Section 3.5 we apply one-loop in α' effective action (not one-string-loop) for the common sector which generalizes (3.1.7)–(3.1.8) and reads [165]

$$S = \frac{1}{2\lambda_s^{D-2}} \int d^Dx \sqrt{-g}\ e^{-\phi} \left\{ R - 2\Lambda + (\nabla\phi)^2 - \frac{1}{12} H^2 \right.$$
$$- \alpha' \lambda_0 \left[R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} - \frac{1}{2} R^{\mu\nu\sigma\rho} H_{\mu\nu\alpha} H_{\sigma\rho}{}^{\alpha} \right.$$
$$\left. \left. + \frac{1}{24} H_{\mu\nu\lambda} H^{\nu}{}_{\rho\alpha} H^{\rho\sigma\lambda} H_{\sigma}{}^{\mu\alpha} - \frac{1}{8} H_{\mu\rho\lambda} H_{\nu}{}^{\rho\lambda} H^{\mu\sigma\alpha} H^{\nu}{}_{\sigma\alpha} \right] + O(\alpha'^2) \right\}, \tag{3.1.11}$$

where $\lambda_0 = -\frac{1}{8}$ for heterotic strings, $-\frac{1}{4}$ for bosonic strings and 0 for superstrings (type I and II); D is the number of spacetime dimensions (either 10 or 26). Thus, in the case of superstrings the leading corrections after the zero-order terms will enter at fourth order in Riemann tensor and will not be investigated. These higher-order terms lead to a significant increase in algebraic complexity. In (3.1.11) ϕ is the dilaton, g the determinant of the metric, Λ the cosmological constant, R the Ricci scalar, $R_{\mu\nu\rho\sigma}$ the Riemann tensor, and $H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]}$ is the axion with $H^2 = H_{\mu\nu\rho}H^{\mu\nu\rho}$ where $B_{\mu\nu}$ is the anti-symmetric tensor potential. The field equations for bosonic strings $\lambda_0 = -\frac{1}{4}$ then take the following form

$$\begin{aligned} R_{\mu\beta} \quad &+ \quad \partial_\mu\partial_\beta\phi - \frac{1}{4}H_{\mu\nu\lambda}H_\beta{}^{\nu\lambda} + \frac{1}{2}\alpha' \left[R_{\mu\nu\sigma\rho}R_\beta{}^{\nu\sigma\rho} - R_\mu{}^{\nu\sigma\rho}H_{\beta\nu\alpha}H_{\sigma\rho}{}^\alpha \right. \\ &\left. + \frac{1}{8}H_{\mu\nu\lambda}H^\nu{}_{\rho\alpha}H^{\rho\sigma\lambda}H_{\sigma\beta}{}^\alpha - \frac{3}{8}H_{\mu\rho\lambda}H_\nu{}^{\rho\lambda}H_\beta{}^{\sigma\alpha}H^\nu{}_{\sigma\alpha}\right] = 0 \end{aligned} \tag{3.1.12}$$

$$R - 2\Lambda + 2\partial_\mu\partial^\mu\phi - \partial_\mu\phi\partial^\mu\phi - \frac{1}{12}H^2 + \frac{1}{4}\alpha' \left[R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} - \right. \tag{3.1.13}$$

$$\left. -\frac{1}{2}R^{\mu\nu\sigma\rho}H_{\mu\nu\alpha}H_{\sigma\rho}{}^\alpha + \frac{1}{24}H_{\mu\nu\lambda}H^\nu{}_{\rho\alpha}H^{\rho\sigma\lambda}H_\sigma{}^{\mu\alpha} - \frac{1}{8}H_{\mu\rho\lambda}H_\nu{}^{\rho\lambda}H^{\mu\sigma\alpha}H^\nu{}_{\sigma\alpha}\right] = 0.$$

These equations are completed by the axion equation of motion

$$\partial_\mu(\delta S/\delta(\partial_\mu B_{\nu\lambda})) = \left[e^{-\phi}\left(H^{\mu\nu\lambda} + \alpha' M^{\mu\nu\lambda}\right)\right]_{;\mu} = 0, \tag{3.1.14}$$

where

$$M^{\mu\nu\lambda} = \frac{3}{2}R^{\mu\nu\sigma\rho}H_{\sigma\rho}{}^\lambda + \frac{1}{4}H_\sigma{}^{\mu\alpha}H^\nu{}_{\rho\alpha}H^{\rho\sigma\lambda} - \frac{3}{4}H^{\mu\sigma\alpha}H_\rho{}^{\nu\lambda}H^\rho{}_{\sigma\alpha}. \tag{3.1.15}$$

The cosmological constant term Λ is related to the dimension of space and to the inverse string tension by [44]

$$\Lambda = \frac{D-10}{3\alpha'} \tag{3.1.16}$$

for superstrings, and

$$\Lambda = \frac{D-26}{3\alpha'} \tag{3.1.17}$$

for bosonic strings.

Our notation so far was that Greek indices μ, ν, ρ etc. were running from $0, \ldots, D-1$ where D was number of spacetime dimensions. We will use this notation except Section 3.4 where we introduce orthonormal frames. The field equations in the string frame which come from the action (3.1.9) are simply the field equations (3.1.12)–(3.1.15) for $\alpha' \to 0$ and read

$$R^{\nu}_{\mu} + \nabla_{\mu}\nabla^{\nu}\phi - \frac{1}{4}H_{\mu\alpha\beta}H^{\nu\alpha\beta} = 0, \tag{3.1.18}$$

$$R - \nabla_{\mu}\phi\nabla^{\mu}\phi + 2\nabla_{\mu}\nabla^{\mu}\phi - \frac{1}{12}H_{\mu\nu\beta}H^{\mu\nu\beta} = 0, \tag{3.1.19}$$

$$\nabla_{\mu}\left(e^{-\phi}H^{\mu\nu\alpha}\right) = \partial_{\mu}\left(e^{-\phi}\sqrt{-g}H^{\mu\nu\alpha}\right) = 0. \tag{3.1.20}$$

These equations can be rewritten in the so-called Einstein frame by the conformal transformation, as follows:

$$g^{S}_{\mu\nu} = \exp\left(\frac{2}{2-D}\phi\right)g^{E}_{\mu\nu}, \tag{3.1.21}$$

where $g^{S}_{\mu\nu}$ is the metric in the string frame, while $g^{E}_{\mu\nu}$ is the metric in the Einstein frame. In the Einstein frame the fundamental string length $\lambda_s = \lambda_s(t)$ (changes with time) while the Planck mass $M_{pl} =$ const. (compare (3.1.10)).

Using (3.1.21) for $D = 4$ the 4-dimensional string frame action (3.1.9) in the Einstein frame reads as

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\ \left\{R + \partial_{\mu}\phi\partial^{\mu}\phi - \frac{1}{12}e^{-2\phi}H^2\right\}, \tag{3.1.22}$$

and the field equations are

$$\tilde{R}^{\nu}_{\mu} - \frac{1}{2}\tilde{g}^{\nu}_{\mu}\tilde{R} = \kappa^2\left(\tilde{T}^{\nu(\phi)}_{\mu} + \tilde{T}^{\nu(H)}_{\mu}\right), \tag{3.1.23}$$

$$\tilde{\nabla}_{\mu}\tilde{\nabla}^{\mu}\phi + \frac{1}{6}e^{-2\phi}\tilde{H}^2 = 0, \tag{3.1.24}$$

$$\tilde{\nabla}_{\mu}\left(e^{-2\phi}\tilde{H}^{\mu\nu\alpha}\right) = \tilde{\partial}_{\mu}\left(e^{-2\phi}\sqrt{-\tilde{g}}\tilde{H}^{\mu\nu\alpha}\right) = 0, \tag{3.1.25}$$

and ($\kappa^2 = 8\pi G, c = 1$)

$$\kappa^2\tilde{T}^{\nu(\phi)}_{\mu} = \frac{1}{2}\left(\tilde{g}^{\lambda}_{\mu}\tilde{g}^{\nu\sigma} - \frac{1}{2}\tilde{g}^{\nu}_{\mu}\tilde{g}^{\lambda\sigma}\right)\tilde{\nabla}_{\lambda}\phi\tilde{\nabla}_{\sigma}\phi, \tag{3.1.26}$$

$$\kappa^2\tilde{T}^{\nu(H)}_{\mu} = \frac{1}{12}e^{-2\phi}\left(3\tilde{H}_{\mu\alpha\beta}\tilde{H}^{\nu\alpha\beta} - \frac{1}{2}\tilde{g}^{\nu}_{\mu}\tilde{H}^2\right). \tag{3.1.27}$$

3.2 Friedmann (pre-big-bang) string cosmologies

The simplest string cosmologies are given for isotropic Friedmann-Robertson-Walker geometries with metric of the form [118]

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-\mathcal{K}r^2} + r^2 d\Omega^2\right), \tag{3.2.1}$$

where $\mathcal{K} = +1, 0, -1$ for spatially closed, flat and open models, respectively. Introducing conformal time coordinate $d\eta = dt/a$, this metric can be rewritten to give

$$ds^2 = a^2(\eta)\left\{-d\eta^2 + d\chi^2 + \left(\frac{\sin\sqrt{\mathcal{K}}\chi}{\sqrt{\mathcal{K}}}\right)^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right\}, \tag{3.2.2}$$

where

$$r = S_{\mathcal{K}}(\chi) \quad = \quad \begin{cases} \sin\chi \quad \mathcal{K} = +1, \\ \chi \quad \mathcal{K} = 0, \\ \sinh\chi \quad \mathcal{K} = -1. \end{cases} \tag{3.2.3}$$

All the physical fields in Friedmann models should be spatially homogeneous (time-dependent only) – their spatial-dependence is not compatible with the symmetric metric (3.2.2).

Isotropic Friedmann cosmology based on the equation (3.2.1) is totally different (in the string frame) from the standard general relativistic cosmology. For $\mathcal{K} = 0$, and $H = 0$ (no axion), one gets from (3.1.18)–(3.1.20) the following solutions for the scale factor a and for the dilaton ϕ

$$\begin{aligned} a(t) &= \mid t \mid^{\pm\frac{1}{\sqrt{3}}}, \\ e^{\phi(t)} &= \mid t \mid^{\pm\sqrt{3}-1}. \end{aligned} \tag{3.2.4}$$

These solutions led to the cosmological scenario which is called *pre-big-bang* cosmology [99, 100, 200]. We discuss its main ideas in due course.

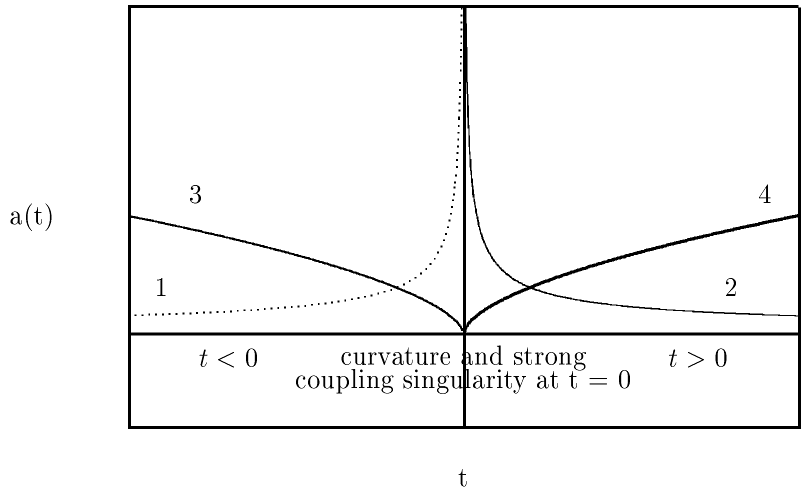

Figure 3.1: Pre-big-bang cosmology. There are four possible types of evolution of the scale factor (the evolution of the dilaton is qualitatively the same) which are commonly called branches. Branches **1** and **2** correspond to '–' sign in (3.2.4) while branches **3** and **4** correspond to '+' sign in (3.2.4). Branches **1**–**3** and **2**–**4** are duality-related. Branch **1** is superinflationary while branch **4** is deflationary and describes usual radiation-dominated evolution. The universe begins with an asymptotically trivial state (minus infinity) of weak coupling and low curvature (Minkowski vacuum), undergoes superinflation driven by the kinetic energy of the dilaton and then reaches curvature and strong coupling singularity at t = 0. After possible 'exit' the universe reaches usual radiation-dominated evolution

It is easy to notice that the solutions (3.2.4) admit a phase of expansion for *negative* times as well as for positive times (with the singularity formally located at $t = 0$). In principle, similar extension of the evolution to negative times can also be made in general relativity but without such a natural connection to the unified theory of gauge interactions and gravity. At $t = 0$, there is a singularity of two types. Firstly, it is a curvature singularity which appears in general relativity as well. Secondly, it is a strong coupling singularity in the sense of string theory, since here string coupling constant $g_s = e^{\frac{\phi}{2}}$ (cf. (1.1.3)) diverges for $t \to 0$.

From (3.2.4) we realize that there are four possible types of the evolution for the scale factor together with four corresponding types of evolution for the dilaton.

These with '–' sign in (3.2.4) will be numbered as **1** and **2** while those with '+' sign in (3.2.4) will be numbered as **3** and **4** (cf. Fig. 3.1). All of them are commonly called branches. Branches **1** and **3** apply for negative times ($t < 0$) while branches **2** and **4** apply for positive times ($t > 0$). The four types of evolution are connected by the underlying symmetry of string theory namely $T - duality$ (also called O(d,d) symmetry [168]) which in the context of isotropic cosmology is called scale factor duality (SFD) [101]. Its mathematical realization is given by the relation which change the scale factor and the dilaton, leaving field equations unchanged, i.e.,

$$a(t) \Longleftrightarrow \frac{1}{a(t)}, \tag{3.2.5}$$

$$\phi(t) \Longleftrightarrow \phi(t) - \ln a^6(t). \tag{3.2.6}$$

SFD relates **1** and **3** or **2** and **4** whose domains are either for $t < 0$ or $t > 0$. However, there is also time-reflection symmetry

$$t \Longleftrightarrow -t, \tag{3.2.7}$$

which together with SFD gives relation beetween **1** and **4** as follows

$$a_1(t) = (-t)^{-\frac{1}{\sqrt{3}}} \Longleftrightarrow t^{\frac{1}{\sqrt{3}}} = \frac{1}{a_4(-t)}. \tag{3.2.8}$$

It is easy to show that for branch **1**

$$\frac{\ddot{a}_1}{a_1} > 0, \tag{3.2.9}$$

which means that it describes inflation which undergoes without a *violation of the energy conditions*. This comes form the fact that there is only *kinetic term* for the dilaton in the action and there is no potential energy at all. So, we have kinetic-energy-driven inflation in opposition to standard inflation [47] which is potential-energy-driven inflation. Such a type of inflation has also been called k-inflation [7], although usually, in the context of the bosonic theory (3.1.9), it is known as *superinflation* or dilaton-driven inflation. It is easy to notice that the branch **4** is deflationary, i.e.,

$$\frac{\ddot{a}_4}{a_4} < 0, \tag{3.2.10}$$

and it describes standard *radiation-dominated* evolution. Branches **1** and **4** are duality-related, though, they are divided by the singularity of curvature and strong coupling.

In the case of nonzero spatial curvature the pre-big-bang solutions are [56]

$$a(\tau) = a_0\sqrt{\frac{\tau^{1+\sqrt{3}\zeta}}{1+\mathcal{K}\tau^2}}, \qquad (3.2.11)$$

$$e^{\phi(t)} = \phi_0\tau^{\sqrt{3}\zeta}, \qquad (3.2.12)$$

$(a_0, \phi_0, \zeta = \text{const.})$ where we have introduced the time coordinate

$$\tau \equiv \begin{cases} \mathcal{K}^{-1//2}|\tan(\mathcal{K}^{1//2}\eta)| & \mathcal{K} > 0, \\ |\eta| & \mathcal{K} = 0, \\ |\mathcal{K}|^{-1//2}|\tanh(|\mathcal{K}|^{1//2}\eta)| & \mathcal{K} < 0. \end{cases} \qquad (3.2.13)$$

This time coordinate diverges for both early and late times η in the models with $\mathcal{K} \geq 0$, but it has the limit $\tau \to |\mathcal{K}|^{-1//2}$ in the models with negative curvature.

In the Einstein frame, the scale factor reads as

$$\tilde{a} = a\exp\left(-\frac{\phi}{2}\right), \qquad (3.2.14)$$

and the evolution of the universe is given by

$$\tilde{a} = \tilde{a_0}\sqrt{\frac{\tau}{1+\mathcal{K}\tau^2}}. \qquad (3.2.15)$$

These models have the curvature singularity for $\eta = 0$, where $\tilde{a} = 0$, and the universe expands for $\eta > 0$ or collapses for $\eta < 0$. Notice, that here closed models recollapse for $\eta = \pm\pi/2$ and in the Einstein frame there is no singularity-free model of the universe. Then, in the Einstein frame, the evolution of the universe is alike the one in general relativity and no 'stringy' effects are visible.

3.3 Kantowski-Sachs string cosmologies

In this Section we consider string cosmologies based on the field equations (3.1.18)–(3.1.20) with Kantowski-Sachs homogeneous geometries (2.3.1). We shall consider models of all three curvatures in the same analysis although only the $\mathcal{K} = +1$ models fall outside the Bianchi classification. The nonzero Ricci tensor components are

$$-R_0^0 = \frac{\ddot{X}}{X} + 2\frac{\ddot{Y}}{Y}, \tag{3.3.1}$$

$$-R_1^1 = \frac{\ddot{X}}{X} + 2\frac{\dot{X}}{X}\frac{\dot{Y}}{Y}, \tag{3.3.2}$$

$$-R_2^2 = -R_3^3 = \frac{\mathcal{K} + \dot{Y}^2}{Y^2} + \frac{\ddot{Y}}{Y} + \frac{\dot{X}}{X}\frac{\dot{Y}}{Y}, \tag{3.3.3}$$

and the scalar curvature is

$$-R = 2\frac{\ddot{X}}{X} + 4\frac{\ddot{Y}}{Y} + 2\frac{\mathcal{K} + \dot{Y}^2}{Y^2} + 4\frac{\dot{X}}{X}\frac{\dot{Y}}{Y}. \tag{3.3.4}$$

Since the metric is spatially homogeneous the dilaton field can only depend on time and we have

$$\nabla_\mu \nabla^\nu \phi = \phi^{;\nu}_{,\mu} + \Gamma^\nu_{\mu\rho}\phi^{,\rho}, \tag{3.3.5}$$

so

$$\nabla_0 \nabla^0 \phi = \ddot{\phi}, \tag{3.3.6}$$

$$\nabla_1 \nabla^1 \phi = \frac{\dot{X}}{X}\dot{\phi}, \tag{3.3.7}$$

$$\nabla_2 \nabla^2 \phi = \nabla_3 \nabla^3 \phi = \frac{\dot{Y}}{Y}\dot{\phi}, \tag{3.3.8}$$

$$\nabla_0 \phi \nabla^0 \phi = \dot{\phi}^2. \tag{3.3.9}$$

3.3.1 Vacuum-dilaton-axion universes – solitonic ansatz

We start this subsection with the introduction of the notation for the three-index axion field H in terms of the pseudoscalar axion field h. Following [57], we define spacetime dual vector $h_{,\beta}$ to H field as

$$H^{\mu\nu\alpha} = e^{\phi}\epsilon^{\mu\nu\alpha\beta}h_{,\beta}. \tag{3.3.10}$$

and

$$\epsilon^{\mu\nu\alpha\beta} = \frac{4!}{\sqrt{-g}}\delta^{\mu}_{[0}\delta^{\nu}_{1}\delta^{\alpha}_{2}\delta^{\beta}_{3]}. \qquad (3.3.11)$$

The equation of motion for the h-field is obtained via the integrability condition $dH = 0$ as

$$\nabla^{\mu}\nabla_{\mu}h + \nabla^{\mu}\phi\nabla_{\mu}h = 0. \qquad (3.3.12)$$

Notice that for spatially-dependent antisymmetric tensor potential, $B_{\mu\nu} = B_{\mu\nu}(x)$, the h field can only depend on time $h = h(t)$, and equation (3.3.12) reads

$$\ddot{h} + \left(\frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y}\right)\dot{h} + \dot{\phi}\dot{h} = 0, \qquad (3.3.13)$$

which integrates to give

$$\dot{h} = -A\frac{e^{-\phi}}{XY^2}. \qquad (3.3.14)$$

So, from (3.3.10), we have ($A =$ const.)

$$H^{123} = -\frac{A}{X^2Y^4S(\theta)}, \qquad (3.3.15)$$

or

$$H_{123} = AS(\theta), \qquad (3.3.16)$$

with $S(\theta)$ given by (2.3.3). With the H field chosen as above, the equation of motion (3.1.20) is satisfied. There is also a trivial solution of (3.3.13), $\dot{h} = 0$, but it corresponds to a constant torsion field. All other components of the H field are zero.

The ansatz (3.3.16) is known as solitonic ansatz [27, 170]. It is equivalent to a spatially-dependent antisymmetric tensor potential $B_{\mu\nu} = B_{\mu\nu}(r,\theta,\psi)$. One can easily calculate that the antisymmetric tensor potential components for (3.3.16) are given by $B_{12} = A\psi\sin\theta, B_{23} = Ar\sin\theta, B_{31} = -A\cos\theta$ for $\mathcal{K} = +1$; $B_{12} = A\psi\sinh\theta, B_{23} = Ar\sinh\theta, B_{31} = A\cosh\theta$ for $\mathcal{K} = -1$ and $B_{12} = A\psi, B_{23} = Ar, B_{31} = A\theta$ for $\mathcal{K} = 0$. With the choice (3.3.16) the field equations (3.1.18) become

$$\frac{\ddot{X}}{X} + 2\frac{\ddot{Y}}{Y} - \ddot{\phi} \;\; = \;\; 0, \qquad (3.3.17)$$

$$\frac{\ddot{X}}{X} + 2\frac{\dot{X}}{X}\frac{\dot{Y}}{Y} - \dot{\phi}\frac{\dot{X}}{X} - \frac{1}{2}\frac{A^2}{X^2Y^4} = 0, \tag{3.3.18}$$

$$\frac{\mathcal{K} + \dot{Y}^2}{Y^2} + \frac{\ddot{Y}}{Y} + \frac{\dot{X}}{X}\frac{\dot{Y}}{Y} - \dot{\phi}\frac{\dot{Y}}{Y} - \frac{1}{2}\frac{A^2}{X^2Y^4} = 0. \tag{3.3.19}$$

The sum of (3.3.17) and (3.3.18) added to twice (3.3.19) gives

$$2\frac{\ddot{X}}{X} + 4\frac{\ddot{Y}}{Y} + 2\frac{\mathcal{K} + \dot{Y}^2}{Y^2} + 4\frac{\dot{X}}{X}\frac{\dot{Y}}{Y} - \ddot{\phi} - \left(\frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y}\right)\dot{\phi} - \frac{3}{2}\frac{A^2}{X^2Y^4} = 0. \tag{3.3.20}$$

The field equation (3.1.19) reads

$$-2\frac{\ddot{X}}{X} - 4\frac{\ddot{Y}}{Y} - 2\frac{\mathcal{K} + \dot{Y}^2}{Y^2} - 4\frac{\dot{X}}{X}\frac{\dot{Y}}{Y} + 2\ddot{\phi} - \dot{\phi}^2 + 2\left(\frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y}\right)\dot{\phi} + \frac{1}{2}\frac{A^2}{X^2Y^4} = 0, \tag{3.3.21}$$

so from the sum of (3.3.20) and (3.3.21) we have

$$\ddot{\phi} - \dot{\phi}^2 + \left(\frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y}\right)\dot{\phi} - \frac{A^2}{X^2Y^4} = 0. \tag{3.3.22}$$

At this stage we introduce a new time coordinate τ via relation

$$dt = XY^2e^{-\phi}d\tau, \tag{3.3.23}$$

and then (3.3.22) becomes

$$\phi_{,\tau\tau} - A^2e^{-2\phi} = 0, \tag{3.3.24}$$

which solves as

$$e^{\phi} = \cosh\alpha\tau + \sqrt{1 - \frac{A^2}{\alpha^2}}\sinh\alpha\tau, \tag{3.3.25}$$

with α constant ($\alpha^2 > A^2$). If we turn off the $H_{\mu\nu\rho}$ field, the solution of (3.3.25) is simply

$$\phi(\tau) = \alpha\tau + \gamma, \tag{3.3.26}$$

with γ a constant, which we may set to zero without loss of generality. A useful relation, implied by (3.3.25), is

$$\phi_{,\tau\tau} + \phi_{,\tau}^2 = \alpha^2. \tag{3.3.27}$$

Using the time coordinate (3.3.23), equations (3.3.17)-(3.3.19) become

$$\begin{aligned} \left(\frac{X_{,\tau}}{X}\right)_{,\tau} + 2\left(\frac{Y_{,\tau}}{Y}\right)_{,\tau} - 2\frac{Y_{,\tau}}{Y}\left(\frac{Y_{,\tau}}{Y} + 2\frac{X_{,\tau}}{X}\right) &+ \\ 2\phi_{,\tau}\left(\frac{X_{,\tau}}{X} + 2\frac{Y_{,\tau}}{Y}\right) - \phi_{,\tau\tau} - \phi_{,\tau}^2 &= 0, \qquad (3.3.28) \\ \left(\frac{X_{,\tau}}{X}\right)_{,\tau} - \frac{1}{2}\phi_{,\tau\tau} &= 0, \qquad (3.3.29) \\ \left(\frac{Y_{,\tau}}{Y}\right)_{,\tau} - \frac{1}{2}\phi_{,\tau\tau} + \mathcal{K}X^2Y^2e^{-2\phi} &= 0. \qquad (3.3.30) \end{aligned}$$

The equations (3.3.28) and (3.3.29) can be rewritten as

$$\begin{aligned} \left(\ln X^2 e^{-\phi}\right)_{,\tau\tau} &= 0, \qquad (3.3.31) \\ \left(\ln Y^2 e^{-\phi}\right)_{,\tau\tau} + 2\mathcal{K}X^2Y^2e^{-2\phi} &= 0. \qquad (3.3.32) \end{aligned}$$

From (3.3.21), we see that the constraint equation (3.3.17) can be rewritten as

$$\frac{\mathcal{K} + \dot{Y}^2}{Y^2} + 2\frac{\dot{X}}{X}\frac{\dot{Y}}{Y} + \frac{1}{2}\dot{\phi}^2 - \dot{\phi}\left(\frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y}\right) - \frac{1}{4}\frac{A^2}{X^2Y^4} = 0, \tag{3.3.33}$$

so, in terms of the time coordinate (3.3.23),

$$\frac{1}{2}\left(\ln X^2e^{-\phi}\right)_{,\tau}\left(\ln Y^2e^{-\phi}\right)_{,\tau} + (\ln Y)_{,\tau}(\ln Y - \phi)_{,\tau} + \mathcal{K}X^2Y^2e^{-2\phi} = \frac{1}{4}A^2e^{-2\phi}. \tag{3.3.34}$$

The solution of (3.3.31) is

$$X^2e^{-\phi} = X_0e^{p\tau}, \tag{3.3.35}$$

with X_0 and p constants. Then, from (3.3.25) and (3.3.35), for $A \neq 0$, we have

$$X(\tau) = \sqrt{X_0}e^{\frac{1}{2}p\tau}\sqrt{\cosh\alpha\tau + \sqrt{1 - \frac{A^2}{\alpha^2}}\sinh\alpha\tau}, \tag{3.3.36}$$

or, for $A = 0$,

$$X(\tau) = \sqrt{X_0} e^{\frac{1}{2}(p+\alpha)\tau}. \tag{3.3.37}$$

The solution of (3.3.32) for the scale factor Y is given by

$$Y(\tau) = \frac{1}{\sqrt{X_0}} e^{-\frac{1}{2}p\tau} \sqrt{\cosh \alpha\tau + \sqrt{1 - \frac{A^2}{\alpha^2}} \sinh \alpha\tau} \sqrt{M(\tau)}, \tag{3.3.38}$$

where $M(\tau) = X^2 Y^2 \exp(-2\phi)$ satisfies

$$(\ln M)_{,\tau\tau} + 2\mathcal{K}M = 0; \tag{3.3.39}$$

hence,

$$\frac{1}{\sqrt{M(\tau)}} = \cosh \beta\tau + \sqrt{1 - \frac{\mathcal{K}}{\beta^2}} \sinh \beta\tau, \tag{3.3.40}$$

and $\beta^2 > \mathcal{K}$. The constraint equation (3.3.34) may be now rewritten in terms of M as

$$\frac{1}{4}\left(\frac{M_{,\tau}}{M}\right)^2 + \mathcal{K}M = \frac{1}{4}\left(\alpha^2 + p^2\right), \tag{3.3.41}$$

which gives the condition

$$\beta^2 = \frac{1}{4}\left(\alpha^2 + p^2\right). \tag{3.3.42}$$

The solutions (3.3.36) and (3.3.38) for the two scale factors X and Y in the string frame can also be rewritten in the Einstein frame as $\tilde{X}$ and $\tilde{Y}$ and they are related via conformal transformation (3.1.21) in the form (3.3.60)–(3.3.62) [170].

3.3.2 Deparametrised vacuum-dilaton universes

From (3.3.25), (3.3.36) and (3.3.38) we realize that without the axion field ($A = 0$) the solutions for ϕ, X and Y reduce to

$$\phi(\tau) = \alpha\tau, \tag{3.3.43}$$

$$X(\tau) = \sqrt{X_0} e^{\frac{1}{2}(\alpha+p)\tau}, \tag{3.3.44}$$

$$Y(\tau) = \frac{1}{\sqrt{X_0}} e^{\frac{1}{2}(\alpha-p)\tau} \tilde{M}, \tag{3.3.45}$$

where $\tilde{M}(\tau)$, the solution of (3.3.39), is given by

$$\tilde{M} = \begin{cases} \beta \cosh^{-1}(\beta\tau + \delta) & \text{for } \mathcal{K} = +1, \\ \exp(\beta\tau + \delta) & \text{for } \mathcal{K} = 0, \\ \beta \sinh^{-1}(\beta\tau + \delta) & \text{for } \mathcal{K} = -1, \end{cases} \tag{3.3.46}$$

where δ is constant, with the constraint given by (3.3.41).

From (3.3.23) and (3.3.43)–(3.3.45) we find that the time parameter in the string frame is

$$t(\tau) = \frac{1}{\sqrt{X_0}} \int e^{-\frac{1}{2}(\alpha - p)\tau} \tilde{M}^2 d\tau. \tag{3.3.47}$$

Hence, this relation is integrable for $\mathcal{K} \neq 0$, provided $\alpha = p$ (that is, from (3.3.42), if $\beta^2 = \alpha^2/2$). In this case we have

$$t(\tau) = \frac{\alpha^2}{2\sqrt{X_0}} \begin{cases} \pm\frac{\sqrt{2}}{\alpha} \tanh\left(\pm\frac{\alpha}{\sqrt{2}}\tau + \delta\right), & \mathcal{K} = +1, \\ \mp\frac{\sqrt{2}}{\alpha} \coth\left(\mp\frac{\alpha}{\sqrt{2}}\tau + \delta\right), & \mathcal{K} = -1. \end{cases} \tag{3.3.48}$$

For $\mathcal{K} = 0$ (3.3.47) is integrable for any α and p. It gives

$$t(\tau) = \frac{1}{X_0 s} \exp(-s\tau - 2\delta), \tag{3.3.49}$$

where

$$s = -\frac{1}{2}(\alpha - p + 4\beta). \tag{3.3.50}$$

After deparametrisation, equations (3.3.43) – (3.3.45) provide a simple solution of (3.3.17)-(3.3.19) for $A = 0$ and $\dot{\phi} = \dot{X}/X$. When $\mathcal{K} \neq 0$, it is given by

$$X(t) = \left(\mathcal{K}\frac{\frac{\alpha}{\sqrt{2}} - t}{\frac{\alpha}{\sqrt{2}} + t}\right)^{\frac{1}{\sqrt{2}}}, \tag{3.3.51}$$

$$Y(t) = \sqrt{\mathcal{K}\left(\frac{\alpha^2}{2} - t^2\right)}, \tag{3.3.52}$$

$$\phi(t) = \ln\left(\mathcal{K}\frac{\frac{\alpha}{\sqrt{2}} - t}{\frac{\alpha}{\sqrt{2}} + t}\right)^{\frac{1}{\sqrt{2}}} \tag{3.3.53}$$

where the time coordinate has the ranges

$$0 \leq t \leq \frac{\alpha^2}{2} \quad \text{for} \quad \mathcal{K} = +1, \tag{3.3.54}$$

$$t \geq \frac{\alpha^2}{2} \quad \text{for} \quad \mathcal{K} = -1. \tag{3.3.55}$$

The volume expansion is given by

$$V(t) = XY^2 = \left[\mathcal{K}\left(\frac{\alpha}{\sqrt{2}} - t\right)\right]^{\frac{\sqrt{2}+1}{\sqrt{2}}} \left(\frac{\alpha}{\sqrt{2}} + t\right)^{\frac{\sqrt{2}-1}{\sqrt{2}}}, \tag{3.3.56}$$

and its evolution is qualitatively the same as that of the scale factor $Y(t)$.

For $\mathcal{K} = 0$ we have

$$X(t) = \sqrt{X_0}\,(X_0 st)^{\frac{\alpha+p}{\alpha-p+4\beta}}, \tag{3.3.57}$$

$$Y(t) = \frac{1}{X_0}\,(X_0 st)^{\frac{\alpha-p+2\beta}{\alpha-p+4\beta}}, \tag{3.3.58}$$

$$\phi(t) = \frac{2\alpha}{\alpha - p + 4\beta} \ln (X_0 st). \tag{3.3.59}$$

For $\mathcal{K} = +1$, the universe starts at a cigar singularity with $X = \infty, Y = 0$ and terminates at a point singularity with $X = Y = 0$, [53]. For $\mathcal{K} = -1$, the universe either starts at $t = \alpha^2/2$ with a point singularity with the ensuing volume expansion going to infinity (with asymptotic value of $X = 1$ for $t \to \infty$), or it starts with infinite volume (with X taken to be equal to one at minus infinity) and collapses to a cigar singularity.

In order to write these solutions in the Einstein frame we need to change the scale factors X and Y, via the conformal transformation (3.3.25) together with the time coordinate t into [16, 170]

$$\tilde{X} = e^{-\frac{\phi}{2}} X, \tag{3.3.60}$$

$$\tilde{Y} = e^{-\frac{\phi}{2}} Y, \tag{3.3.61}$$

$$d\tilde{t} = e^{-\frac{\phi}{2}} dt = \left(\mathcal{K}\frac{\frac{\alpha}{\sqrt{2}} - t}{\frac{\alpha}{\sqrt{2}} + t}\right)^{\frac{1}{2\sqrt{2}}} dt. \tag{3.3.62}$$

The calculations for the $\mathcal{K} = 0$ (Bianchi I) case have been given in [57].

3.3.3 Vacuum-dilaton-axion universes – elementary ansatz

In this subsection we consider the space-dependent pseudoscalar axion field $h = h(x)$ called an elementary ansatz. In the case of Kantowski-Sachs metric (2.3.1) it means that $h = h(r, \theta, \psi)$. Using (3.3.10) this requires $H^{123} = \epsilon^{1230} e^{\phi} \partial_0 h = 0$ and only the H_{0ij} $(i, j, k = 1, 2, 3)$ components of the axion field H can be non-zero. It seems that space-dependent h should correspond to a time-dependent pseudoscalar axion potential $B_{\mu\nu} = B_{\mu\nu}(t)$ as it was shown for Bianchi models in [20, 57]. This relies on the fact of antisymmetry of the field strength H and requires

$$H_{ijk} = 0 \quad \text{and} \quad H_{0ij} \neq 0. \tag{3.3.63}$$

We will show, however, that in our formulation $B_{\mu\nu}$ is not only the function of time but also the function of spatial coordinates. This is due to the fact that we do not use orthonormal frames in this Section 3.3 – the Kantowski-Sachs metric (2.3.1) is written down in a holonomic (coordinate) frame.

Assuming that $h = h(x)$, the equation of motion (3.3.12) for the Kantowski-Sachs metric (2.3.1) reads

$$\frac{1}{X^2}\partial_1^2 h + \frac{1}{Y^2}\partial_2^2 h + \frac{1}{Y^2 S^2(\theta)}\partial_3^2 h + \frac{1}{Y^2} C(\theta)\partial_2 h = 0, \tag{3.3.64}$$

where

$$C(\theta) = \begin{cases} \cot\theta & \text{for} \quad \mathcal{K} = +1, \\ 0 & \text{for} \quad \mathcal{K} = 0, \\ \coth\theta & \text{for} \quad \mathcal{K} = -1. \end{cases} \tag{3.3.65}$$

There are some conditions which must be satisfied if the Kantowski-Sachs geometry is to admit axion field H into the field equations (3.1.18)–(3.1.20). One is that the off-diagonal components of $H_{\mu\lambda\sigma}H^{\nu\lambda\sigma}$ in (3.1.19) should vanish and we have the condition

$$g^{jj} g^{mm} H_{iom} H_{jom} = 0 \quad (i \neq j, \text{no sum}), \tag{3.3.66}$$

which means that *only one* of H_{012}, H_{023}, or H_{013} may be non-zero.

The same condition of vanishing off-diagonal components of the H field can be expressed in terms of the pseudoscalar axion h as follows

$$e^{2\phi} g^{jk} \partial_i h \partial_k h = 0 \quad i \neq j (\text{no sum over } i \text{ and } j). \tag{3.3.67}$$

This means that only one of the three $\partial_i h$ may be non-zero. The solutions of the equation of motion (3.3.64) which satisfy the condition (3.3.67) are as follows

$$\begin{aligned}
\partial_1 h &= D = const., \partial_2 h = \partial_3 h = 0, && (3.3.68)\\
\partial_3 h &= B = const., \partial_1 h = \partial_2 h = 0, && (3.3.69)\\
\partial_2 h &= \frac{E}{S(\theta)}, \partial_1 h = \partial_3 h = 0, && (3.3.70)
\end{aligned}$$

with E constant and $S(\theta)$ given by (2.3.3).

Suppose we choose $H_{012} \neq 0$. From the equation of motion (3.1.20) we obtain

$$H^{012} = \frac{Be^{\phi}}{XY^2 S(\theta)}, \tag{3.3.71}$$

with B constant, and $S(\theta)$ given by (2.3.1), so

$$H_{012} = -\frac{BXe^{\phi}}{S(\theta)}. \tag{3.3.72}$$

One can easily check that the integrability condition is fulfilled, since $\partial_3 H_{012} = 0$. However, (3.1.18) gives

$$H^2 \equiv H_{\mu\nu\lambda} H^{\mu\nu\lambda} = -6B^2 \frac{e^{2\phi} Y^2}{S^2(\theta)} \tag{3.3.73}$$

and there is explicit dependence on the spatial coordinate θ, which means that the ansatz $H_{012} \neq 0$ is inconsistent with the geometries under consideration.

The next possibility is $H_{013} \neq 0$. This means that

$$H^{013} = \frac{Ce^{\phi}}{XY^2 S(\theta)} \tag{3.3.74}$$

and

$$H_{013} = Ee^{\phi} X S(\theta), \tag{3.3.75}$$

with E constant. Then

$$H^2 = \frac{E^2 e^{2\phi}}{Y^2}, \tag{3.3.76}$$

which depends only on time. However, in this case we see that the integrability condition $\partial_2 H_{013} = 0$ is not fulfilled, so the choice $H_{013} \neq 0$ is also impossible.

The last possibility is given by

$$H_{023} = De^{\phi} \frac{Y^2}{X} S(\theta) = Y^4 S^2(\theta) H^{023}, \tag{3.3.77}$$

where D is constant. This time-integrability condition $\partial_1 H_{023} = 0$ is fulfilled and

$$H^2 = 6D^2 \frac{e^{2\phi}}{X^2}, \tag{3.3.78}$$

which depends only on the time coordinate, as required. Thus,

$$H_{023} \neq 0 \tag{3.3.79}$$

provides the only consistent choice. This naturally gives the space dependence of the antisymmetric tensor potential

$$B_{23} = B_{23}(t, \theta) \tag{3.3.80}$$

because $H_{023} = \partial_0 B_{23}$ (due to the gauge transformation we can eliminate all the components B_{0i} [57]), so $B_{23} = DS(\theta) \int Y^2 X^{-1} e^{\phi} dt$ and $B_{12} = B_{13} = 0$. This also shows that H_{123} component of the axion field must vanish ($H_{312} = \partial_3 B_{12} = 0, H_{231} = \partial_2 B_{13} = 0, H_{123} = \partial_1 B_{23}(t, \theta) = 0$. The final conclusion is that our ansatz (3.3.77) is correct.

The conclusion about $B_{\mu\nu}$ being both time and space-dependent apparently contradicts the results of [20, 57] where spatial-dependence of the pseudoscalar axion field h was associated to the time-dependence of axion potential $B_{\mu\nu}$. This difference appears here because we worked in coordinate frames rather than in orthonormal frames of [20, 27]. In general one can use coordinate frames in the calculations for Bianchi I model [57], but it did not really matter because the basis forms in type I are just $\sigma^1 = dx^1, \sigma^2 = dx^2, \sigma^3 = dx^3$, and do not involve any spatial dependence. One could of course elaborate the problem in orthonormal frames [186] coming to the same conclusion as in the coordinate frames. The only quantity which should be homogeneous (time-dependent) for homogeneous geometry is the energy-momentum tensor expressible in terms of axion H or the pseudoscalar axion field h (cf. (3.1.18)–(3.1.19)) which is the case for our choice (3.3.79).

Using (3.3.77) and (3.3.78), the field equations (3.1.18) become

$$\frac{\ddot{X}}{X} + 2\frac{\ddot{Y}}{Y} - \ddot{\phi} = -\frac{D^2 e^{2\phi}}{2X^2}, \tag{3.3.81}$$

$$\frac{\ddot{X}}{X} + 2\frac{\dot{X}}{X}\frac{\dot{Y}}{Y} - \dot{\phi}\frac{\dot{X}}{X} = 0, \tag{3.3.82}$$

$$\frac{\mathcal{K} + \dot{Y}^2}{Y^2} + \frac{\ddot{Y}}{Y} + \frac{\dot{X}}{X}\frac{\dot{Y}}{Y} - \dot{\phi}\frac{\dot{Y}}{Y} = -\frac{D^2 e^{2\phi}}{2X^2}. \tag{3.3.83}$$

The sum of (3.3.81) and (3.3.82) together with twice (3.3.83) gives

$$2\frac{\ddot{X}}{X} + 4\frac{\ddot{Y}}{Y} + 2\frac{\mathcal{K} + \dot{Y}^2}{Y^2} + 4\frac{\dot{X}}{X}\frac{\dot{Y}}{Y} - \ddot{\phi} - \left(\frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y}\right)\dot{\phi} - \frac{3}{2}\frac{D^2 e^{2\phi}}{X^2} = 0. \tag{3.3.84}$$

The equation (3.1.19) is

$$-2\frac{\ddot{X}}{X} - 4\frac{\ddot{Y}}{Y} - 2\frac{\mathcal{K} + \dot{Y}^2}{Y^2} - 4\frac{\dot{X}}{X}\frac{\dot{Y}}{Y} + 2\ddot{\phi} - \dot{\phi}^2 + 2\left(\frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y}\right)\dot{\phi} - \frac{1}{2}\frac{D^2 e^{2\phi}}{X^2} = 0, \tag{3.3.85}$$

so from the sum of (3.3.82) and (3.3.83) we have

$$\ddot{\phi} - \dot{\phi}^2 + \left(\frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y}\right)\dot{\phi} - \frac{D^2 e^{2\phi}}{X^2} = 0. \tag{3.3.86}$$

Using the time coordinate (3.3.23) equation (3.3.86) becomes

$$\phi_{,\tau\tau} + D^2 Y^4 = 0. \tag{3.3.87}$$

Equations (3.3.81)–(3.3.83) now become

$$\begin{aligned}\left(\frac{X_{,\tau}}{X}\right)_{,\tau} + 2\left(\frac{Y_{,\tau}}{Y}\right)_{,\tau} - 2\frac{Y_{,\tau}}{Y}\left(\frac{Y_{,\tau}}{Y} + 2\frac{X_{,\tau}}{X}\right) & \\ +\ 2\phi_{,\tau}\left(\frac{X_{,\tau}}{X} + 2\frac{Y_{,\tau}}{Y}\right) - \frac{3}{2}\phi_{,\tau\tau} - \phi^2_{,\tau} &= 0, \qquad (3.3.88)\\ \left(\frac{X_{,\tau}}{X}\right)_{,\tau} &= 0, \qquad (3.3.89)\\ \left(\frac{Y_{,\tau}}{Y}\right)_{,\tau} - \frac{1}{2}\phi_{,\tau\tau} + \mathcal{K}X^2Y^2e^{-2\phi} &= 0, \qquad (3.3.90)\end{aligned}$$

and (3.3.88) and (3.3.89) can be rewritten as

$$\begin{aligned}(\ln X)_{,\tau\tau} &= 0, \qquad (3.3.91)\\ \left(\ln Y^2 e^{-\phi}\right)_{,\tau\tau} + 2\mathcal{K}X^2Y^2e^{-2\phi} &= 0. \qquad (3.3.92)\end{aligned}$$

The solution of (3.3.91) is

$$X(\tau) = \exp{(r\tau + s)}, \qquad (3.3.93)$$

with r and s constants. For $k = 0$ it is also possible to solve (3.3.92) to obtain

$$Y(\tau) = \exp\{\frac{1}{2}\left[\phi(\tau) + m\tau + n\right]\}, \qquad (3.3.94)$$

with m and n constants. Using (3.3.93)–(3.3.94) we may solve (3.3.87)–(3.3.90) for Y and ϕ to give

$$\begin{aligned}Y(\tau) &= 2^{\frac{1}{4}}\sqrt{\frac{b}{D}}\left[\cosh b\sqrt{2}\tau\right]^{-\frac{1}{2}}, \qquad (3.3.95)\\ \phi(\tau) &= \phi_0 - m\tau - \ln\cosh b\sqrt{2}\tau, \qquad (3.3.96)\end{aligned}$$

with ϕ_0 constant, and

$$b^2 = m(m + 2r). \qquad (3.3.97)$$

Using (3.3.23) we have

$$\tau(t) = A + \ln t^{\frac{1}{r+m}}, \qquad (3.3.98)$$

where

$$A = \frac{1}{r+m}\left[\ln\frac{D(r+m)}{b\sqrt{2}} - s - \phi_0\right]. \tag{3.3.99}$$

Finally, the solution of (3.3.87) – (3.3.90) in terms of the cosmic time t in the string frame is given by

$$X(t) = t^{\frac{r}{r+m}}, \tag{3.3.100}$$

$$Y(t) = 2^{\frac{1}{4}}\sqrt{\frac{b}{D}}\left(t_0 t^w + \frac{1}{t_0}t^{-w}\right)^{-\frac{1}{2}}, \tag{3.3.101}$$

$$e^{\phi(t)} = e^{-ma}t^{-\frac{m}{r+m}}\left(t_0 t^w + \frac{1}{t_0}t^{-w}\right)^{-1}, \tag{3.3.102}$$

where $t_0 = \exp b\sqrt{2}A$ and $w = b\sqrt{2}/(r+m)$ are constants.

This is an axisymmetric subcase of the Bianchi type I axion-dilaton solution [57].

Some special solutions for the $\mathcal{K} \neq 0$ cases can be given by solving (3.3.87) for Y; that is, by taking

$$Y^2 = \frac{1}{D}\left(-\phi_{,\tau\tau}\right)^{\frac{1}{2}}, \tag{3.3.103}$$

and substituting into (3.3.90), to obtain

$$\left[\ln\left\{\left(-\phi_{,\tau\tau}\right)^{\frac{1}{2}} D^{-1}e^{-\phi}\right\}\right]_{,\tau\tau} + 2\mathcal{K}D^{-1}\left(-\phi_{,\tau\tau}\right)^{\frac{1}{2}} e^{-2\phi(\tau)+2r\tau+2s} = 0. \tag{3.3.104}$$

We now seek solutions of the form

$$\phi(\tau) = \alpha^2\ln\tau + \beta\tau + \epsilon, \tag{3.3.105}$$

with α^2, β, and ϵ constants. From the field equations (3.3.87) – (3.3.90) we have

$$X(\tau) = \exp(r\tau + s), \tag{3.3.106}$$

$$\phi(\tau) = \frac{1}{2}\ln\tau + r\tau + \epsilon, \tag{3.3.107}$$

$$Y(\tau) = \pm 2^{-\frac{1}{4}}D^{-\frac{1}{2}}\tau^{-\frac{1}{2}}, \tag{3.3.108}$$

with the constraint

$$e^{2(\epsilon - s)} = \pm \frac{4\mathcal{K}}{3D\sqrt{2}}, \tag{3.3.109}$$

which, because $D > 0$, means that we take the plus sign for $\mathcal{K} = +1$ and the minus sign for $\mathcal{K} = -1$ universes. Using (3.3.23) and (3.3.106) – (3.3.108), we write

$$t(\tau) = \mp D^{-1}\sqrt{2}e^{s-\epsilon}\tau^{-\frac{1}{2}}. \tag{3.3.110}$$

After deparametrisation, our solutions (3.3.106) – (3.3.108) give

$$X(t) \propto \exp\left(\frac{r}{t^2} + s\right), \tag{3.3.111}$$

$$Y(t) \propto t, \tag{3.3.112}$$

$$\phi(t) \propto \ln t + \frac{const.}{t^2}. \tag{3.3.113}$$

3.3.4 Axion field in homogeneous cosmologies

To conclude, in this section we have considered the common sector effective superstring equations for a Kantowski-Sachs background spacetime. We have included the full bosonic spectrum of fields, with the graviton $g_{\mu\nu}$, dilaton ϕ, and the axion H. We have considered two forms of ansätze for the axion. In terms of the pseudoscalar axion field h, they correspond to a dependence of it on either the time or space coordinates alone.

For the time-dependent case, $h = h(t)$, we have found an exact parametric solution of the field equations given in subsection 3.3.2. In such a case the axion field behaves effectively as a stiff-fluid distributed homogeneously over space. These solutions were also discussed in the context of the scalar-tensor cosmologies by using slightly different parametrization in [170].

We also find that, for vanishing axion $H = 0$, there is a deparametrised exact solution for $\dot{\phi} = \dot{X}/X$. We have discussed this solution in Section 3.3.3. It appears to be the most interesting Kantowski-Sachs solution in which one could study the duality problem.

For the spatially-dependent case, $h = h(r, \theta, \psi)$, we find that there is *only one possible* form for the axion field in spatially homogeneous closed universes of Kantowski-Sachs type. Its 3-form strength can have just one nonzero component, H_{023}, which distributes the field along the two spatial directions on the 2-sphere S^2. This component depends both on time and space and leads to space and time dependence of the only nonzero component of the tensor potential, $B_{23} = B_{23}(t, \theta)$. This is expected because we are working in coordinate frames rather than in orthonormal frames [20, 27]. In effect, there is an anisotropic stress in the universe. For such a dilaton-axion anisotropic cosmology we have written down the field equations and have found some new solutions. In the zero-curvature case we recover the axisymmetric Bianchi I solutions [57]. These results provide, in particular, a new type of closed universe in string cosmology.

The fact that only one component of the axion field H under elementary ansatz (3.3.77) is admissible to Kantowski-Sachs geometry imposes a potential concern for late-time isotropization of such vacuum-axion-dilaton models, which seems to be one of the basic requirement to satisfy observational constraint on the present-day isotropy of the universe [139]. On the other hand, the axion field H is one of the 'stringy' modes and, despite what one usually does when talking about pre-big-bang scenario (cf. section 3.2), it cannot be neglected in cosmological considerations. Of course the solitonic ansatz (3.3.16) puts no restrictions on the late-time isotropization of the models and is fully acceptable – though not elementary (see the considerations for the Einstein-Maxwell equations in [35]).

3.4 Mixmaster string cosmologies

3.4.1 Bianchi IX homogeneous geometries – solitonic ansatz

Bianchi IX universe model or Mixmaster model is the most general of the homogeneous models and it possesses $SO(3)$ group of symmetry acting on spaces of homogeneity.

It is well known that in the vacuum BIX homogeneous cosmology one approaches the initial singularity chaotically [30, 62, 63]. An infinite number of oscillations of the orthogonal scale factors occurs in general on any finite interval of proper time including the singularity at $t = 0$. These oscillations are created by the 3-curvature anisotropy of the spacetime and are intrinsically general relativistic in origin [11, 12]. Physically, the propagation of homogeneous gravitational waves alters the curvature of spacetime along the direction of propagation, so that their non-linear back-reaction on the curvature reverses the direction of propagation.

In the presence of a massless scalar field (or stiff matter) the situation changes. Only a finite number of spacetime oscillations can occur before the evolution is changed into a state in which all directions shrink monotonically to zero as the curvature singularity is reached and the oscillatory behaviour disappears [29]. In this section we want to investigate the oscillatory approach to singularity in BIX effective-action homogeneous string cosmologies with both axion and dilaton fields.

Because of a complicated form of Bianchi IX metric in holonomic frames in this section we will work in orthonormal frames [186] which means that in the field equations (3.1.18)–(3.1.20) we replace Greek indices α, β, μ, ν by Latin ones $i, j, k, l = 0, 1, 2, 3$. Explicit relations between all the geometric quantities of the Bianchi IX models in coordinate (holonomic) frames versus orthonormal frames are given in Appendix A.

Covariant derivatives in the field equations (3.1.18)–(3.1.20) are formed with respect to the Bianchi type IX metric which, in the string frame, reads as [17]

$$ds^2 = dt^2 - a^2(t)(\sigma^1)^2 - b^2(t)(\sigma^2)^2 - c^2(t)(\sigma^3)^2, \tag{3.4.1}$$

where the orthonormal forms $\sigma^1, \sigma^2, \sigma^3$ are given by

$$\sigma^1 = \cos\psi d\theta + \sin\psi \sin\theta d\varphi, \tag{3.4.2}$$

$$\sigma^2 = \sin\psi d\theta - \cos\psi \sin\theta d\varphi, \tag{3.4.3}$$

$$\sigma^3 = d\psi + \cos\theta d\varphi, \tag{3.4.4}$$

and the angular coordinates ψ, θ, φ span the following ranges (compare (2.3.99) in which an axially-symmetric form of (3.4.1) was applied)

$$0 \le \psi \le 4\pi, \quad 0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi. \tag{3.4.5}$$

In the Einstein frame (cf. (3.1.21)) the metric is

$$d\tilde{s}^2 = d\tilde{t}^2 - \tilde{a}^2(t)(\sigma^1)^2 - \tilde{b}^2(t)(\sigma^2)^2 - \tilde{c}^2(t)(\sigma^3)^2, \tag{3.4.6}$$

and

$$\begin{aligned} d\tilde{t} &= e^{\frac{-\phi}{2}} dt, & (3.4.7) \\ \tilde{a}_i &= e^{\frac{-\phi}{2}} a_i, & (3.4.8) \end{aligned}$$

where $\tilde{a}_i = \{\tilde{a}, \tilde{b}, \tilde{c},\}$ and $a_i = \{a, b, c\}$.

The nonzero Ricci tensor components in the orthonormal frame σ^i are (an overdot means a derivative with respect to the synchronous coordinate time t) [30]

$$-R^0_0 = \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c}, \tag{3.4.9}$$

$$-R^1_1 = \frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{a}}{a}\frac{\dot{c}}{c} - \frac{1}{2a^2b^2c^2}\left[\left(b^2 - c^2\right)^2 - a^4\right], \tag{3.4.10}$$

$$-R^2_2 = \frac{\ddot{b}}{b} + \frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{b}}{b}\frac{\dot{c}}{c} - \frac{1}{2a^2b^2c^2}\left[\left(a^2 - c^2\right)^2 - b^4\right], \tag{3.4.11}$$

$$-R^3_3 = \frac{\ddot{c}}{c} + \frac{\dot{a}}{a}\frac{\dot{c}}{c} + \frac{\dot{b}}{b}\frac{\dot{c}}{c} - \frac{1}{2a^2b^2c^2}\left[\left(a^2 - b^2\right)^2 - c^4\right], \tag{3.4.12}$$

and the Ricci scalar reads

$$\begin{aligned} -R &= 2\frac{\ddot{a}}{a} + 2\frac{\ddot{b}}{b} + 2\frac{\ddot{c}}{c} + 2\frac{\dot{a}}{a}\frac{\dot{b}}{b} + 2\frac{\dot{a}}{a}\frac{\dot{c}}{c} + 2\frac{\dot{b}}{b}\frac{\dot{c}}{c} \\ &\quad - \frac{1}{2a^2b^2c^2}\left[a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2a^2c^2\right]. \end{aligned} \tag{3.4.13}$$

Since the model under consideration is homogeneous the dilaton field can only depend on time and so we have (compare Appendix A)

$$\begin{aligned} \nabla_0\nabla^0\phi &= \ddot{\phi}, & (3.4.14) \\ \nabla_i\nabla^i\phi &= \frac{\dot{a}_i}{a_i}\dot{\phi}, & (3.4.15) \end{aligned}$$

$$\nabla_0\phi\nabla^0\phi = \dot{\phi}^2. \tag{3.4.16}$$

As for the axion field, in a similar way as for Kantowski-Sachs models of Section 3.3, one should discuss possible application of both solitonic and elementary ansätze into Bianchi IX geometry. Let us start with solitonic ansatz which essentially requires spatial dependence of the antisymmetric tensor potential $B_{\alpha\beta} = B_{\alpha\beta}(x)$ and time dependence of the pseudoscalar axion field h. If that is the case, then the h-field equation of motion (in a coordinate frame – see Appendix A) (3.3.12) reads as

$$\ddot{h} + \left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right)\dot{h} + \dot{\phi}\dot{h} = 0, \tag{3.4.17}$$

which implies that

$$\dot{h} = -A\frac{e^{-\phi}}{abc}, \tag{3.4.18}$$

so from (3.3.10) we have ($\bar{1}, \bar{2}, \bar{3}$ are coordinate frame indices – see Appendix A)

$$H^{\bar{1}\bar{2}\bar{3}} = -\frac{A}{a^2b^2c^2 sin\theta}, \tag{3.4.19}$$

or

$$H_{\bar{1}\bar{2}\bar{3}} = Asin\theta, \tag{3.4.20}$$

and $H^2 = -6A^2/a^2b^2c^2$. With the H field chosen as above the equation of motion (3.1.20) is easily fulfilled. There is also a trivial solution of (3.3.13), $\dot{h} = 0$, but it corresponds to a constant torsion field.

The ansatz (3.4.20) can be expressed in an orthonormal frame as follows [27, 57]

$$H = H_{123}e^1 \wedge e^2 \wedge e^3 = \frac{A}{abc}e^1 \wedge e^2 \wedge e^3 = A\sigma^1 \wedge \sigma^2 \wedge \sigma^3, \tag{3.4.21}$$

where A is constant and

$$e^i = a_i \sigma^i. \tag{3.4.22}$$

Since

$$(H_i^j)^2 \equiv H_{ikl}H^{jkl}, \tag{3.4.23}$$

and

$$H^2 \equiv H_{ikl}H^{ikl}, \tag{3.4.24}$$

we have

$$(H_0^0)^2 = 0, \tag{3.4.25}$$

$$(H_1^1)^2 = (H_2^2)^2 = (H_3^3)^2 = -2\frac{A^2}{a^2b^2c^2}, \tag{3.4.26}$$

$$H^2 = -6\frac{A^2}{a^2b^2c^2}. \tag{3.4.27}$$

Let us introduce now the special case of axial symmetry $a = b$ for which the orthonormal forms (3.4.2)–(3.4.4) are given by (compare (2.3.99) [35]

$$\sigma^1 = d\theta, \tag{3.4.28}$$

$$\sigma^2 = \sin\theta d\varphi, \tag{3.4.29}$$

$$\sigma^3 = d\psi + \cos\theta d\varphi, \tag{3.4.30}$$

Now we check a possible validity of the elementary ansatz into the Bianchi IX geometry.

For the time-independent pseudoscalar axion field $h = h(x)$ (time-dependent antisymmetric tensor potential $B_{\alpha\beta} = B_{\alpha\beta}(t)$) the equation of motion (3.3.12) reads

$$\partial^\alpha\partial_\alpha h + \Gamma^\alpha_{\beta\alpha}\partial^\beta h = 0, \tag{3.4.31}$$

which for the metric (3.4.1) reads as

$$g^{\bar{1}\bar{1}}\partial_{\bar{1}}^2 h + g^{\bar{2}\bar{2}}\partial_{\bar{2}}^2 h + g^{\bar{3}\bar{3}}\partial_{\bar{3}}^2 h + g^{\bar{1}\bar{3}}\partial_{\bar{1}}\partial_{\bar{3}} h + g^{\bar{2}\bar{2}}\cot\theta\partial_{\bar{2}} h = 0. \tag{3.4.32}$$

For simplicity, let us introduce

$$(H_\mu^\nu)^2 = H_{\mu\alpha\beta}H^{\nu\alpha\beta} = -2e^{2\phi}\left(\delta_\mu^\nu g^{\rho\varepsilon} - \delta_\mu^\rho g^{\nu\varepsilon}\right)\partial_\varepsilon h\partial_\rho h, \tag{3.4.33}$$

$$H^2 = H_{\mu\alpha\beta}H^{\mu\alpha\beta}. \tag{3.4.34}$$

The non-zero components of these quantities (the energy-momentum tensor) which are used in the field equations (3.1.18)–(3.1.20) for the metric components (A.27) are given by

$$
\begin{aligned}
(H^{\bar{0}}_{\bar{0}})^2 &= -2e^{2\phi}\Big(g^{\bar{1}\bar{1}}\partial_{\bar{1}}h\partial_{\bar{1}}h + 2g^{\bar{1}\bar{3}}\partial_{\bar{1}}h\partial_{\bar{3}}h \\
&+ g^{\bar{2}\bar{2}}\partial_{\bar{2}}h\partial_{\bar{2}}h + g^{\bar{3}\bar{3}}\partial_{\bar{3}}h\partial_{\bar{3}}h\Big), \qquad (3.4.35) \\
(H^{\bar{1}}_{\bar{1}})^2 &= -2e^{2\phi}\Big(g^{\bar{1}\bar{3}}\partial_{\bar{1}}h\partial_{\bar{3}}h + g^{\bar{2}\bar{2}}\partial_{\bar{2}}h\partial_{\bar{2}}h + g^{\bar{3}\bar{3}}\partial_{\bar{3}}h\partial_{\bar{3}}h\Big), \qquad (3.4.36) \\
(H^{\bar{2}}_{\bar{2}})^2 &= -2e^{2\phi}\Big(g^{\bar{1}\bar{1}}\partial_{\bar{1}}h\partial_{\bar{1}}h + 2g^{\bar{1}\bar{3}}\partial_{\bar{1}}h\partial_{\bar{3}}h + g^{\bar{3}\bar{3}}\partial_{\bar{3}}h\partial_{\bar{3}}h\Big), \qquad (3.4.37) \\
(H^{\bar{3}}_{\bar{3}})^2 &= -2e^{2\phi}\Big(g^{\bar{1}\bar{1}}\partial_{\bar{1}}h\partial_{\bar{1}}h + g^{\bar{1}\bar{3}}\partial_{\bar{1}}h\partial_{\bar{3}}h + g^{\bar{2}\bar{2}}\partial_{\bar{2}}h\partial_{\bar{2}}h\Big), \qquad (3.4.38) \\
(H^{\bar{1}}_{\bar{3}})^2 &= 2e^{2\phi}\Big(g^{\bar{1}\bar{1}}\partial_{\bar{1}}h\partial_{\bar{3}}h + g^{\bar{1}\bar{3}}\partial_{\bar{3}}h\partial_{\bar{3}}h\Big), \qquad (3.4.39)
\end{aligned}
$$

and

$$H^2 = -6e^{2\phi}\Big(g^{\bar{1}\bar{1}}\partial_{\bar{1}}h\partial_{\bar{1}}h + 2g^{\bar{1}\bar{3}}\partial_{\bar{1}}h\partial_{\bar{3}}h + g^{\bar{2}\bar{2}}\partial_{\bar{2}}h\partial_{\bar{2}}h + g^{\bar{3}\bar{3}}\partial_{\bar{3}}h\partial_{\bar{3}}h\Big). \qquad (3.4.40)$$

With the choice

$$\partial_{\bar{3}}h = E\sin\theta \neq 0, \qquad \partial_{\bar{1}}h = \partial_{\bar{2}}h = 0, \qquad (3.4.41)$$

the field equations remain homogeneous and

$$
\begin{aligned}
(H^{\bar{0}}_{\bar{0}})^2 &= (H^{\bar{1}}_{\bar{1}})^2 = (H^{\bar{2}}_{\bar{2}})^2 = 2\frac{E^2e^{2\phi}}{a^2}, \qquad (3.4.42) \\
(H^{\bar{3}}_{\bar{3}})^2 &= 0, \qquad (3.4.43) \\
(H^{\bar{1}}_{\bar{3}})^2 &= 2\frac{E^2e^{2\phi}}{a^2}\cos\theta, \qquad (3.4.44) \\
H^2 &= 6\frac{E^2e^{2\phi}}{a^2}. \qquad (3.4.45)
\end{aligned}
$$

The equation of motion (3.4.32) becomes

$$g^{\bar{3}\bar{3}}\partial_{\varphi}(E\sin\theta) = 0, \qquad (3.4.46)$$

and it is satisfied. Finally, from (3.3.10), we have

$$H^{\bar{0}\bar{1}\bar{2}} = \frac{Ee^{\phi}}{a^2c}, \qquad (3.4.47)$$

$$H_{\bar{0}\bar{1}\bar{2}} = Ece^{\phi}, \qquad (3.4.48)$$

and $H^2 = 6Ee^{2\phi}/a^2$. This obeys the axion equation of motion (3.1.20) but is in contradiction with the axisymmetry condition for the Ricci components (A.37) since $R^2_2 = R^3_3$ there while $(H^{\bar{3}}_{\bar{3}})^2 = 0$ and $(H^{\bar{2}}_{\bar{2}})^2 \neq 0$ here.

A possible ansatz which would fulfil the axisymmetry condition would be

$$\partial_{\bar{1}} h = \frac{Ba(t)\sin\theta}{\sqrt{a(t)^2\sin^2\theta + c(t)^2\cos^2\theta}}, \tag{3.4.49}$$

but it leads to both time and space dependences of the pseudoscalar axion field, $h = h(t, \psi, \theta)$, and does not fulfil the equation of motion (3.3.12) (nor (3.4.32) which is obtained for $h = h(x)$).

One could also try to add

$$\partial_{\bar{2}} h = D = \text{const.}, \tag{3.4.50}$$

to the nonzero component (3.4.49), but again this does not fulfil the equation (3.4.32).

The final conclusion is that one is not able to impose the axion field even in axisymmetric BIX models despite the fact that there is a distinguished direction in the model (which is different from electromagnetic field case – see [35]). The reason seems to be that even in the axisymmetric case there is still $SO(3)$ symmetry group present and we are only adding an additional symmetry $SO(2)$ which does not cancel the former one, giving the total symmetry $SO(3) \otimes SO(2)$ rather than just $SO(2)$.

Because of that in the whole Section 3.4 devoted to Bianchi IX models, we will only use solitonic ansatz (3.4.21).

3.4.2 Oscillatory approach to a singularity

The ansatz (3.4.21) is, in fact, equivalent to the inclusion of a time-dependent pseudoscalar axion field $h = h(t)$ [57] as in subsection 3.3.1. We are going to investigate the possible emergence of chaos in such models. Using (3.4.9)–(3.4.16) and (3.4.25)–(3.4.27) the field equations (3.1.18) in the string frame read ($\dot{} = d/dt$)

$$\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}-\ddot{\phi}=0, \quad (3.4.51)$$

$$\frac{\ddot{a}}{a}+\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{\dot{a}}{a}\frac{\dot{c}}{c}-\frac{\dot{a}}{a}\dot{\phi}-\frac{1}{2a^2b^2c^2}\left[\left(b^2-c^2\right)^2-a^4\right]-\frac{1}{2}\frac{A^2}{a^2b^2c^2}=0, \quad (3.4.52)$$

$$\frac{\ddot{b}}{b}+\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{\dot{b}}{b}\frac{\dot{c}}{c}-\frac{\dot{b}}{b}\dot{\phi}-\frac{1}{2a^2b^2c^2}\left[\left(a^2-c^2\right)^2-b^4\right]-\frac{1}{2}\frac{A^2}{a^2b^2c^2}=0, \quad (3.4.53)$$

$$\frac{\ddot{c}}{c}+\frac{\dot{a}}{a}\frac{\dot{c}}{c}+\frac{\dot{b}}{b}\frac{\dot{c}}{c}-\frac{\dot{c}}{c}\dot{\phi}-\frac{1}{2a^2b^2c^2}\left[\left(a^2-b^2\right)^2-c^4\right]-\frac{1}{2}\frac{A^2}{a^2b^2c^2}=0. \quad (3.4.54)$$

Now (3.1.19), with the help of (3.4.51), gives [27]

$$-2\left(\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{\dot{a}}{a}\frac{\dot{c}}{c}+\frac{\dot{b}}{b}\frac{\dot{c}}{c}\right)+2\dot{\phi}\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}+\frac{\dot{c}}{c}\right)-\dot{\phi}^2$$
$$+\frac{1}{2a^2b^2c^2}\left[a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2\right]+\frac{1}{2}\frac{A^2}{a^2b^2c^2}=0. \quad (3.4.55)$$

Using (3.4.55), together with the sum of (3.4.52)–(3.4.54), we have the dilaton equation of motion

$$\ddot{\phi}-(\dot{\phi})^2+\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}+\frac{\dot{c}}{c}\right)\dot{\phi}-\frac{A^2}{a^2b^2c^2}=0. \quad (3.4.56)$$

In the Einstein frame, the field equations (3.1.23) become [57] ($' = d/d\tilde{t}$)

$$-\frac{\tilde{a}''}{\tilde{a}}-\frac{\tilde{b}''}{\tilde{b}}-\frac{\tilde{c}''}{\tilde{c}} = \frac{1}{2}\phi'^2+\frac{1}{2}A^2\frac{e^{-2\phi}}{\tilde{a}^2\tilde{b}^2\tilde{c}^2}, \quad (3.4.57)$$

$$\frac{\tilde{a}''}{\tilde{a}}+\frac{\tilde{a}'}{\tilde{a}}\frac{\tilde{b}'}{\tilde{b}}+\frac{\tilde{a}'}{\tilde{a}}\frac{\tilde{c}'}{\tilde{c}} = \frac{1}{2\tilde{a}^2\tilde{b}^2\tilde{c}^2}\left[\left(\tilde{b}^2-\tilde{c}^2\right)^2-\tilde{a}^4\right], \quad (3.4.58)$$

$$\frac{\tilde{b}''}{\tilde{b}}+\frac{\tilde{a}'}{\tilde{a}}\frac{\tilde{b}'}{\tilde{b}}+\frac{\tilde{b}'}{\tilde{c}}\frac{\tilde{c}'}{\tilde{c}} = \frac{1}{2\tilde{a}^2\tilde{b}^2\tilde{c}^2}\left[\left(\tilde{a}^2-\tilde{c}^2\right)^2-\tilde{b}^4\right], \quad (3.4.59)$$

$$\frac{\tilde{c}''}{\tilde{c}}+\frac{\tilde{c}'}{\tilde{c}}\frac{\tilde{b}'}{\tilde{b}}+\frac{\tilde{a}'}{\tilde{a}}\frac{\tilde{c}'}{\tilde{c}} = \frac{1}{2\tilde{a}^2\tilde{b}^2\tilde{c}^2}\left[\left(\tilde{a}^2-\tilde{b}^2\right)^2-\tilde{c}^4\right], \quad (3.4.60)$$

and (3.1.24) now becomes

$$\phi'' + \phi' \left(\frac{\tilde{a}'}{\tilde{a}} + \frac{\tilde{b}'}{\tilde{b}} + \frac{\tilde{c}'}{\tilde{c}} \right) = A^2 \frac{e^{-2\phi}}{\tilde{a}^2 \tilde{b}^2 \tilde{c}^2}. \tag{3.4.61}$$

A new time coordinate is introduced to simplify the field equations by defining [27, 57]

$$d\eta = \frac{e^{\phi}}{abc} dt = \frac{1}{\tilde{a}\tilde{b}\tilde{c}} d\tilde{t}. \tag{3.4.62}$$

First, notice that the string-frame equation (3.4.56) then simplifies to

$$\phi_{,\eta\eta} - A^2 e^{-2\phi} = 0, \tag{3.4.63}$$

where $(\ldots)_{,\eta} = d(\ldots)/d\eta$. The Einstein-frame (3.4.61), with the time coordinate $\tilde{t}$, gives the same result, (3.4.63). The solution of (3.4.63) is

$$e^{\phi} = \cosh \Lambda M \eta + \sqrt{1 - \frac{A^2}{\Lambda^2 M^2}} \sinh \Lambda M \eta, \tag{3.4.64}$$

with M, Λ constant ($\Lambda^2 M^2 > A^2$). A useful relation, implied by (3.4.64) is

$$\phi_{,\eta\eta} + \phi_{,\eta}^2 = \Lambda^2 M^2. \tag{3.4.65}$$

Let us introduce new forms for the scale factors

$$a = e^{\alpha} \qquad b = e^{\beta} \qquad c = e^{\gamma}, \tag{3.4.66}$$

and

$$\tilde{a} = e^{\tilde{\alpha}} \qquad \tilde{b} = e^{\tilde{\beta}} \qquad \tilde{c} = e^{\tilde{\gamma}}, \tag{3.4.67}$$

so, from (3.4.8)

$$\tilde{\alpha} = \alpha - \phi/2 \qquad \tilde{\beta} = \beta - \phi/2 \qquad \tilde{\gamma} = \gamma - \phi/2, \tag{3.4.68}$$

and $dt = \exp(\phi/2) d\tilde{t}$ in (3.4.62). The field equations (3.4.51)–(3.4.54) in the string frame take the form

$$(\alpha + \beta + \gamma)_{,\eta\eta} - M^2 = 2\left(\alpha_{,\eta}\beta_{,\eta} + \alpha_{,\eta}\gamma_{,\eta} + \beta_{,\eta}\gamma_{,\eta}\right)$$

$$- \quad 2\left(\alpha_{,\eta}+\beta_{,\eta}+\gamma_{,\eta}\right)\phi_{,\eta}, \tag{3.4.69}$$

$$2e^{2\phi}\alpha_{,\eta\eta} = \left(b^2-c^2\right)^2 - a^4 + A^2, \tag{3.4.70}$$

$$2e^{2\phi}\beta_{,\eta\eta} = \left(a^2-c^2\right)^2 - b^4 + A^2, \tag{3.4.71}$$

$$2e^{2\phi}\gamma_{,\eta\eta} = \left(a^2-b^2\right)^2 - c^4 + A^2. \tag{3.4.72}$$

The equations (3.4.70)–(3.4.72), using (3.4.63), can be rewritten to give

$$(-\phi+2\alpha)_{,\eta\eta} = \left[\left(b^2-c^2\right)^2 - a^4\right]e^{-2\phi}, \tag{3.4.73}$$

$$(-\phi+2\beta)_{,\eta\eta} = \left[\left(a^2-c^2\right)^2 - b^4\right]e^{-2\phi}, \tag{3.4.74}$$

$$(-\phi+2\gamma)_{,\eta\eta} = \left[\left(a^2-b^2\right)^2 - c^4\right]e^{-2\phi}. \tag{3.4.75}$$

Notice that there is no explicit dependence of the axion, A, in these equations (3.4.73)–(3.4.75).

On the other hand, using (3.4.62), (3.4.57)–(3.4.60) in the Einstein frame reduce to

$$2\left(\tilde{\alpha}_{,\eta}\tilde{\beta}_{,\eta}+\tilde{\alpha}_{,\eta}\tilde{\gamma}_{,\eta}+\tilde{\beta}_{,\eta}\tilde{\gamma}_{,\eta}\right) = \left(\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma}\right)_{,\eta\eta} + \frac{1}{2}M^2, \tag{3.4.76}$$

$$2\tilde{\alpha}_{,\eta\eta} = \left(\tilde{b}^2-\tilde{c}^2\right)^2 - \tilde{a}^4, \tag{3.4.77}$$

$$2\tilde{\beta}_{,\eta\eta} = \left(\tilde{a}^2-\tilde{c}^2\right)^2 - \tilde{b}^4, \tag{3.4.78}$$

$$2\tilde{\gamma}_{,\eta\eta} = \left(\tilde{a}^2-\tilde{b}^2\right)^2 - \tilde{c}^4. \tag{3.4.79}$$

These equations (except for the constant M which appears in (3.4.76)) have exactly the same form as the standard BIX equations of general relativity [30]. Equations (3.4.76)–(3.4.79) can be explicitly transformed into (3.4.73)–(3.4.75) by using (3.4.66)–(3.4.68).

In order to discuss possible emergence of chaos we consider suitable initial conditions expressed in terms of the Kasner parameters.

Einstein Frame

The Kasner solutions are obtained as approximate solutions of the equations (3.4.57)–(3.4.60) when the right-hand sides (describing the curvature anisotropies) are neglected. In the Einstein frame, in terms of $\tilde{t}$-time, they are

$$\begin{aligned} \tilde{a} &= \tilde{a}_0 \tilde{t}^{\tilde{p}_1}, \\ \tilde{b} &= \tilde{b}_0 \tilde{t}^{\tilde{p}_2}, \\ \tilde{c} &= \tilde{c}_0 \tilde{t}^{\tilde{p}_3}, \end{aligned} \tag{3.4.80}$$

while ($A^2 \leq \tilde{\Lambda}^2 M^2$)

$$\phi = \ln \tilde{d}_0 + \frac{M}{\tilde{\Lambda}} \ln |\tilde{t}|, \tag{3.4.81}$$

for $A = 0$, and

$$\phi = \ln \tilde{d}_0 + \ln \left[\frac{1}{2} \left(1 + \sqrt{1 - \frac{A^2}{\tilde{\Lambda} M^2}} \right) |\tilde{t}|^{\frac{M}{\tilde{\Lambda}}} + \frac{1}{2} \left(1 - \sqrt{1 - \frac{A^2}{\tilde{\Lambda} M^2}} \right) |\tilde{t}|^{-\frac{M}{\tilde{\Lambda}}} \right], \tag{3.4.82}$$

for $A \neq 0$, where

$$\tilde{\Lambda} = \tilde{a}_0 \tilde{b}_0 \tilde{c}_0, \qquad \tilde{d}_0 = \text{const.} \tag{3.4.83}$$

From (3.4.58)–(3.4.60), irrespective to the presence of the axion term A, we have the following algebraic conditions on the Kasner indices, $\tilde{p}_i$:

$$\sum_{i=1}^{3} \tilde{p}_i = 1, \tag{3.4.84}$$

and, from (3.4.57),

$$\sum_{i=1}^{3} \tilde{p}_i^2 = 1 - \frac{M^2}{2\tilde{\Lambda}^2}, \tag{3.4.85}$$

which is exactly the case considered first by Belinskii and Khalatnikov [29] and describes general relativity minimally coupled to a scalar field, or, alternatively general reltivity with stiff-fluid source.

In terms of η-time from (3.4.77)–(3.4.79)

$$\begin{aligned} \tilde{\alpha} &= \tilde{\Lambda}\tilde{p}_1\eta + \tilde{r}_1, \\ \tilde{\beta} &= \tilde{\Lambda}\tilde{p}_2\eta + \tilde{r}_2, \\ \tilde{\gamma} &= \tilde{\Lambda}\tilde{p}_3\eta + \tilde{r}_3, \end{aligned} \tag{3.4.86}$$

and the constraint equation (3.4.76) becomes

$$\tilde{p}_1\tilde{p}_2 + \tilde{p}_1\tilde{p}_3 + \tilde{p}_2\tilde{p}_3 = \frac{1}{4}\frac{M^2}{\tilde{\Lambda}^2}. \tag{3.4.87}$$

This constraint is equivalent to the constraints (3.4.84)–(3.4.85). In order to get (3.4.80) from (3.4.86) we need to use (3.4.62) to relate $\tilde{t}$-time with η-time, i.e.,

$$\eta = \tilde{\Lambda}^{-1}\ln\tilde{t} + \text{const}. \tag{3.4.88}$$

When we approach singularity $\eta \to -\infty$ ($\tilde{t} \to 0$) we cannot neglect all the terms on the right-hand side of (3.4.77)–(3.4.79) since one of the Kasner indices in (3.4.87) can be negative. We assume $\tilde{p}_1 \equiv -|\tilde{p}_1| < 0$, which means that $\tilde{a}(\eta) \gg \tilde{b}(\eta) \gg \tilde{c}(\eta)$, so (3.4.76)–(3.4.79) are approximated by

$$\begin{aligned} \tilde{\alpha}_{,\eta\eta} &= -\frac{1}{2}e^{4\tilde{\alpha}}, \\ \tilde{\beta}_{,\eta\eta} &= \frac{1}{2}e^{4\tilde{\alpha}}, \\ \tilde{\gamma}_{,\eta\eta} &= \frac{1}{2}e^{4\tilde{\alpha}}. \end{aligned} \tag{3.4.89}$$

Far away from singularity the approximate axisymmetric Kasner regime is fulfilled, but it is broken by the $\tilde{a}^2$ term when we approach singularity, so Kasner solutions (3.4.86) will apply for $\eta \to \infty$ ($\tilde{t} \to \infty$); suitable solutions of (3.4.89) which fulfil (3.4.86) are

$$\begin{aligned} \tilde{\alpha}(\eta) &= -\frac{1}{2}\ln\left[\frac{1}{2|\tilde{p}_1|\tilde{\Lambda}}\cosh\left(-2|\tilde{p}_1|\tilde{\Lambda}\eta\right)\right], \\ \tilde{\beta}(\eta) &= \frac{1}{2}\ln\left[\frac{1}{2|\tilde{p}_1|\tilde{\Lambda}}\cosh\left(-2|\tilde{p}_1|\tilde{\Lambda}\eta\right)\right] + \tilde{\Lambda}\left(-|\tilde{p}_1| + \tilde{p}_2\right)\eta, \end{aligned} \tag{3.4.90}$$

$$\tilde{\gamma}(\eta) = \frac{1}{2}\ln\left[\frac{1}{2|\tilde{p_1}|\tilde{\Lambda}}\cosh\left(-2|\tilde{p_1}|\tilde{\Lambda}\eta\right)\right] + \tilde{\Lambda}\left(-|\tilde{p_1}| + \tilde{p_3}\right)\eta.$$

This choice necessarily requires $\tilde{p_1} < 0$ and of course, one has the same solutions for $\tilde{p_1} > 0$, but in such a case our initial conditions would have to be taken at $\eta \to \infty$ rather than at $\eta \to -\infty$. It does not change the physics of the problem and was assumed in [114, 123], but we prefer to follow the spirit of [30].

In the limit $\eta \to -\infty$ $(\tilde{t} \to 0)$ from (3.4.90) we have

$$\begin{aligned} \tilde{\alpha}(\eta) &= \tilde{\Lambda}|\tilde{p_1}|\eta = -\tilde{\Lambda}\tilde{p_1}\eta, \\ \tilde{\beta}(\eta) &= \tilde{\Lambda}\left(\tilde{p_2} - 2|\tilde{p_1}|\right)\eta = \tilde{\Lambda}\left(\tilde{p_2} + 2\tilde{p_1}\right)\eta, \\ \tilde{\gamma}(\eta) &= \tilde{\Lambda}\left(\tilde{p_2} - 2|\tilde{p_1}|\right)\eta = \tilde{\Lambda}\left(\tilde{p_3} + 2\tilde{p_1}\right)\eta, \end{aligned} \tag{3.4.91}$$

which means that one Kasner epoch (3.4.86) with indices $\tilde{p_i}$ is replaced by another Kasner epoch with indices given by (3.4.91). By virtue of (3.4.62)

$$\eta = \tilde{\Lambda}'^{-1}\ln\tilde{t} + \text{const.}, \tag{3.4.92}$$

and

$$\begin{aligned} \tilde{a} &= \tilde{a}'_0\tilde{t}^{\tilde{p}'_1}, \\ \tilde{b} &= \tilde{b}'_0\tilde{t}^{\tilde{p}'_2}, \\ \tilde{c} &= \tilde{c}'_0\tilde{t}^{\tilde{p}'_3}, \end{aligned} \tag{3.4.93}$$

where

$$\begin{aligned} \tilde{p}'_1 &= -\frac{|\tilde{p_1}|}{1-2|\tilde{p_1}|} = \frac{\tilde{p_1}}{1+2\tilde{p_1}}, \\ \tilde{p}'_2 &= \frac{\tilde{p_2} - 2|\tilde{p_1}|}{1-2|\tilde{p_1}|} = \frac{\tilde{p_2}+2\tilde{p_1}}{1+2\tilde{p_1}}, \\ \tilde{p}'_3 &= \frac{\tilde{p_3} - 2|\tilde{p_1}|}{1+2\tilde{p_1}} = \frac{\tilde{p_3}+2\tilde{p_1}}{1+2\tilde{p_1}}, \end{aligned} \tag{3.4.94}$$

and

$$\tilde{\Lambda}'\tilde{t} = \tilde{a}\tilde{b}\tilde{c}\tilde{d}, \qquad \frac{\tilde{\Lambda}'}{\tilde{\Lambda}} = 1 + 2\tilde{p_1},$$

$$\sum_{i=1}^{3} \tilde{p}'_i = 1, \qquad \sum_{i=1}^{3} \tilde{p}'^2_i = 1 - \frac{1}{2}\frac{M^2}{\tilde{\Lambda}'^2}. \tag{3.4.95}$$

As we can see from (3.4.95), the axion does not really influence these asymptotic solutions.

One can easily show that

$$-\sqrt{\frac{2}{3}} \leq \frac{M}{\tilde{\Lambda}\sqrt{2}} \leq \sqrt{\frac{2}{3}} \tag{3.4.96}$$

and, after ordering the Kasner indices by $\tilde{p}_1 \leq \tilde{p}_2 \leq \tilde{p}_3$, we require

$$\begin{aligned} -1/3 &\leq \tilde{p}_1 \leq 1/3, \\ 0 &\leq \tilde{p}_2 \leq 2/3, \\ 1/3 &\leq \tilde{p}_3 \leq 1. \end{aligned} \tag{3.4.97}$$

This means that, unlike the vacuum case, all the Kasner indices can be positive and, as in the analysis of [29], the final situation is that the universe inevitably reaches a monotonic stage of evolution towards singularity (see also the discussion of Section 3.4.4). This follows since the terms of the type $e^{4\tilde{\alpha}}, e^{4\tilde{\beta}}, e^{4\tilde{\gamma}}$ decrease if $\eta \to -\infty$ $(t \to 0)$, and they do not allow a transition into another Kasner epoch to take place. This is just monotonic evolution like in isotropic Friedman case. By means of (3.4.80) and (3.4.82), which lead to the same relations between the Kasner indices (3.4.84)–(3.4.85), one can extend this conclusion into the axion-dilaton cosmology. The presence of the axion field cannot change the fate of the universe near to a singularity and the universe finally reaches a monotonic stage of evolution with all three scale factors tending to zero as $t \to 0$. Thus, there is *no chaos* in BIX homogeneous string cosmology in the Einstein frame. Finally, we note that there is an isotropic Friedmann limit here, provided all the Kasner indices are equal with $\tilde{p}_1 = \tilde{p}_2 = \tilde{p}_3 = 1/3$.

String Frame

For the string frame we can follow the discussion given in [114] (see also [24, 123]).

In terms of t-time, Kasner solutions of the system (3.4.51)–(3.4.54), for $A = 0$, are given by

$$\begin{aligned} a &= a_0 t^{p_1}, \\ b &= b_0 t^{p_2}, \\ c &= c_0 t^{p_3}, \\ e^{-\phi} &= d_0 t^{p_4}, \end{aligned} \tag{3.4.98}$$

or, alternatively, the last of these conditions (3.4.98) reads as

$$\phi(t) = -\ln d_0 - p_4 \ln|t|, \quad \text{and} \quad p_4 = -M. \tag{3.4.99}$$

After putting (3.4.98) into (3.4.56) we have

$$p_4 = 1 - \sum_{i=1}^{3} p_i, \tag{3.4.100}$$

and from (3.4.51) we have

$$\sum_{i=1}^{3} p_i^2 = 1, \tag{3.4.101}$$

so (3.4.99) can be rewritten to give

$$\phi(t) = -\ln d_0 + (\sum_{i=1}^{3} p_i - 1) \ln t. \tag{3.4.102}$$

The last condition means, for instance, that the choice $p_i = (-1/3, 2/3, 2/3)$ gives $\sum p_i = 1$ and leads to a vanishing or constant dilaton ($M = 0$), i.e., to general relativity. (Other permutations of the three p_i with the signs changed are allowed). Thus, the difference between general relativity and string theory depends on the "fourth Kasner index" $p_4 = -M$.

Now, from (3.4.73)–(3.4.75) (one can see that $\Lambda p_i = 2\tilde{\Lambda}\tilde{p}_i$ here)

$$\begin{aligned} \alpha &= \phi/2 + \Lambda(p_1/2)\eta + r_1/2 = \phi/2 + \tilde{\Lambda}\tilde{p}_1\eta + \tilde{r}_1, \\ \beta &= \phi/2 + \Lambda(p_2/2)\eta + r_2/2 = \phi/2 + \tilde{\Lambda}\tilde{p}_2\eta + \tilde{r}_2, \\ \gamma &= \phi/2 + \Lambda(p_3/2)\eta + r_3/2 = \phi/2 + \tilde{\Lambda}\tilde{p}_3\eta + \tilde{r}_3, \end{aligned} \tag{3.4.103}$$

and from (3.4.69)

$$p_1p_2 + p_1p_3 + p_2p_3 = \frac{M^2}{\Lambda^2}. \qquad (3.4.104)$$

The condition (3.4.104), in fact, can be obtained from (3.4.100)–(3.4.101). In order to get (3.4.98) we need to use (3.4.62), i.e.,

$$\eta = \Lambda^{-1}\ln t + \text{const.}, \qquad \Lambda = a_0b_0c_0d_0. \qquad (3.4.105)$$

From the conditions (3.4.100)–(3.4.101) on the p_i, we have necessarily that [29]

$$-1-\sqrt{3} \le M \le -1+\sqrt{3}. \qquad (3.4.106)$$

However, the domain of M given by (3.4.106) covers the whole duality-related region since in the case of a Kasner regime the duality symmetry (cf. subsection 3.4.3) simply means that we change

$$p_1 \to -p_1,\ p_2 \to -p_2,\ p_3 \to -p_3,\ M \to M - p_1 - p_2 - p_3 = -(M+2). \qquad (3.4.107)$$

Having given (3.4.107), we can delineate the two duality-related domains of Kasner indices as follows:

$$-1-\sqrt{3} \le M \le -1,\ -\frac{1}{\sqrt{3}} \le p_1 \le \sqrt{\frac{2}{3}},\ -\frac{2}{3} \le p_2 \le \frac{1}{2}\sqrt{\frac{2}{3}},\ -1 \le p_3 \le -\frac{1}{2}\sqrt{\frac{2}{3}}, \qquad (3.4.108)$$

and

$$-1 \le M \le -1+\sqrt{3},\ -\sqrt{\frac{2}{3}} \le p_1 \le \frac{1}{\sqrt{3}},\ -\frac{1}{2}\sqrt{\frac{2}{3}} \le p_2 \le \frac{2}{3},\ \frac{1}{2}\sqrt{\frac{2}{3}} \le p_3 \le 1. \qquad (3.4.109)$$

Of course, for $-1-\sqrt{3} \le M \le -1$, the indices are ordered so that $p_3 \le p_2 \le p_1$; while, for $-1 \le M \le -1+\sqrt{3}$, they are ordered as $p_1 \le p_2 \le p_3$.

From (3.4.108)–(3.4.109), we can draw some interesting conclusions. First, that the vacuum general relativity case $M = 0$ (with $-1/3 \le p_1 \le 0, 0 \le p_2 \le 2/3, 2/3 \le p_3 \le 1$) is dual to the case $M = -2$ (with $-1 \le p_1 \le -2/3, -2/3 \le p_2 \le 0, 0 \le p_3 \le 1/3$), while the case $M = -1$ is 'self-dual' in M giving $M = -1$ again, although the $p_i's$ change. Second, the $M = -1$ plane is the dividing plane for the duality-related range of the parameters which are defined by

$$
\begin{aligned}
-\sqrt{\frac{2}{3}} &\le p_1 \le -\frac{1}{2}\sqrt{\frac{2}{3}}, \\
-\frac{1}{2}\sqrt{\frac{2}{3}} &\le p_2 \le \frac{1}{2}\sqrt{\frac{2}{3}}, \\
\frac{1}{2}\sqrt{\frac{2}{3}} &\le p_3 \le \sqrt{\frac{2}{3}}.
\end{aligned}
$$

The isotropic Friedmann solution is given when we choose $M = -1 + \sqrt{3}, p_1 = p_2 = p_3 = 1/\sqrt{3}$, while its dual with $M = -1 - \sqrt{3}$, $p_1 = p_2 = p_3 = -1/\sqrt{3}$ does not appear in the vacuum general relativity case. This shows that one can generally have two different types of change in the Kasner indices, and in M (effectively, a fourth Kasner index). There are oscillations as in general relativistic vacuum or stiff fluid cases [29, 30], but there are also duality-related exchanges of indices. We will discuss some points related to duality that refer to the work of [207, 208].

There are eight possible permutations of the Kasner indices in which some are equal to $-1/\sqrt{3}$ or $1/\sqrt{3}$, with suitable M. These are the first two quadruples $(p_1, p_2, p_3, M) = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}, -1+\sqrt{3}), (-1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3}, -1 -\sqrt{3})$, plus another three pairs with $(-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}, -1+(\sqrt{3}/3)), (1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3}, -1-(\sqrt{3}/3))$ and the permutations which are dual to each other. One can easily show that these last three pairs of cases are duality related and describe exactly the transitions from one Kasner epoch to another given by (3.4.107). Another interesting set of 'self-dual' ($M = -1$) combinations which describe LRS (Locally Rotationally Symmetric) Kasner solutions are given by $(\sqrt{2/3}, -1/\sqrt{6}, -1/\sqrt{6}, -1), (-\sqrt{2/3}, 1/\sqrt{6}, 1/\sqrt{6}, -1)$ and their permutations. Other interesting duality-related combinations are $(-1/3, 2/3, 2/3, 0)$, $(1/3, -2/3, -2/3, -2)$ and $(0, 1/\sqrt{2}, -1/\sqrt{2}, -1), (0, -1/\sqrt{2}, 1/\sqrt{2}, -1)$. The last two give the so-called Taub points (flat Minkowski spacetime) [207, 208].

The question arises whether the changes of the Kasner indices (chaotic and duality-related) have something in common. In order to answer this question one should try to find a suitable parametrization of the Kasner indices analogous to the one given in [29] for the stiff-fluid case. We have checked that such a parametrization does not cover the right range of the indices given by (3.4.108) unless we multiply p_1 by $\sqrt{3}$. After some searching, we have found the following parametrization of the indices which has advantages which we will explain in due course.

The u-parametrization we use is defined as follows

$$\begin{aligned} \bar{p}_1 &= \frac{2u}{1+u^2}, \\ \bar{p}_2 &= \frac{1}{2}\frac{1}{1+u^2}\Big[(1+M)(1+u^2) - 2u + P^+(M,u)\Big], \\ \bar{p}_3 &= \frac{1}{2}\frac{1}{1+u^2}\Big[(1+M)(1+u^2) - 2u - P^+(M,u)\Big], \end{aligned} \tag{3.4.110}$$

where u is constant and

$$P^+(M,u) = \sqrt{\left(1-2M-M^2\right)\left(1+u^2\right)^2 + 4u\left(M+1\right)\left(1+u^2\right) - 12u^2}, \tag{3.4.111}$$

and to achieve $p_1 < 0$ we need $u < 0$. Alternatively, we have

$$\begin{aligned} p_1 &= -\frac{2u}{1+u^2}, \\ p_2 &= \frac{1}{2}\frac{1}{1+u^2}\Big[(1+M)(1+u^2) + 2u + P^-(M,u)\Big], \\ p_3 &= \frac{1}{2}\frac{1}{1+u^2}\Big[(1+M)(1+u^2) + 2u - P^-(M,u)\Big], \end{aligned} \tag{3.4.112}$$

where

$$P^-(M,u) = \sqrt{\left(1-2M-M^2\right)\left(1+u^2\right)^2 - 4u\left(M+1\right)\left(1+u^2\right) - 12u^2}, \tag{3.4.113}$$

and for $p_1 < 0$ we need $u > 0$. These are very general transformations that include all duality-related cases (3.4.108)–(3.4.109) and the vacuum general relativity case when $M = 0$. In the general relativity case ($M = 0$) these parametrizations can be written down in terms of the one suggested in [207, 208] with $p_1 = (1/3)(1-2\cos\psi), p_{2,3} = (1/3)(1+\cos\psi \pm \sin\psi)$ with a suitable choice of $\cos\psi = (1/2)(u^2-6u+1)(u^2+1)$ in our case.

There are many possible transformations of the general homographic type $u \to (au+b)/(cu+d)$ or some more general types related to Padé or other rational approximants [8] which may be useful in the discussion of Mixmaster oscillations. First, notice that the transformation of the type

$$u \to \frac{1}{u}, \tag{3.4.114}$$

gives

$$\begin{aligned} \bar{p}_i(M,1/u) &= \bar{p}_i(M,u), \\ p_i(M,1/u) &= p_i(M,u), \end{aligned} \tag{3.4.115}$$

where $i = 1,2,3$, unless we take negative value of the root of $P^{\pm}$ after applying the transformation. On the other hand using

$$u \to -\frac{1}{u}, \tag{3.4.116}$$

changes parametrization (3.4.110) into (3.4.112), i.e.,

$$\bar{p}_i(M,-1/u) = p_i(M,u) = -\bar{p}_i(M,u), \tag{3.4.117}$$

$$p_i(M,-1/u) = \bar{p}_i(M,u) = -p_i(M,u). \tag{3.4.118}$$

Second, one can consider the transformation

$$u \to -u, \tag{3.4.119}$$

which results in the same rules as (3.4.115). However, this transformation is not enough to describe duality relations between the considered solutions. As one can see, the duality-reflecting transformations should be of the type

$$u \to -u, \quad M \to -(M+2), \tag{3.4.120}$$

or, alternatively

$$u \to -\frac{1}{u}, \quad M \to -(M+2). \tag{3.4.121}$$

From (3.4.120) and (3.4.121) we see that

$$P^{\pm}(-M-2,-u) = P^{\pm}(-M-2,-1/u) = P^{\pm}(M,u), \tag{3.4.122}$$

and

$$\begin{aligned} \bar{p}_1(-u) &= -\bar{p}_1(u), \\ \bar{p}_2(-M-2,-u) &= -\bar{p}_3(M,u), \\ \bar{p}_3(-M-2,-u) &= -\bar{p}_2(M,u); \end{aligned} \tag{3.4.123}$$

similarly,

$$\begin{aligned} p_1(-u) &= -p_1(u), \\ p_2(-M-2,-u) &= -p_3(M,u), \\ p_3(-M-2,-u) &= -p_2(M,u). \end{aligned} \tag{3.4.124}$$

We notice that the duality in (3.4.123)–(3.4.124) would entail the exchange of the Kasner indices p_2 and p_3.

If we assume that $a \gg b \gg c$ in (3.4.73)–(3.4.74) then we obtain the following set of approximating equations (for $A = 0$)

$$\begin{aligned} \alpha_{,\eta\eta} &= -\frac{1}{2}e^{4\alpha}e^{-2\Lambda M\eta}, \\ \beta_{,\eta\eta} &= \frac{1}{2}e^{4\alpha}e^{-2\Lambda M\eta}, \\ \gamma_{,\eta\eta} &= \frac{1}{2}e^{4\alpha}e^{-2\Lambda M\eta}, \\ \phi_{,\eta\eta} &= 0, \end{aligned} \tag{3.4.125}$$

together with the constraint (3.4.69). The Kasner solutions in terms of η-time are given by

$$\begin{aligned} \alpha(\eta) &= \Lambda p_1\eta + \text{const.}, \\ \beta(\eta) &= \Lambda p_2\eta + \text{const.}, \\ \gamma(\eta) &= \Lambda p_3\eta + \text{const.}, \\ \phi(\eta) &= \Lambda M\eta + \text{const.} \end{aligned} \tag{3.4.126}$$

However, these solutions are not directly obtained by using the relations (3.4.68) between the Einstein frame and string-frame scale factors. This is very important in the case when the axion field is taken into account and will be discussed below.

The solutions of equations (3.4.125) which fulfil the above conditions, (3.4.126), in the limit $\eta \to \infty$ ($t \to \infty$, i.e., far away from singularity), provided $p_1 = -|p_1| < 0$ and $2p_1 - M = -2|p_1| - M < 0$, can be chosen to be

$$\begin{aligned} \alpha(\eta) &= -\frac{1}{2}\ln\left(\frac{1}{\Lambda(2p_1 - M)}\cosh\Lambda(2p_1 - M)\eta\right) + \frac{1}{2}\Lambda M\eta, \\ \beta(\eta) &= \frac{1}{2}\ln\left(\frac{1}{\Lambda(2p_1 - M)}\cosh\Lambda(2p_1 - M)\eta\right) \\ &+ (p_1 + p_2 - M)\,\Lambda\eta + \frac{1}{2}\Lambda M\eta, \\ \gamma(\eta) &= \frac{1}{2}\ln\left(\frac{1}{\Lambda(2p_1 - M)}\cosh\Lambda(2p_1 - M)\eta\right) \\ &+ (p_1 + p_3 - M)\,\Lambda\eta + \frac{1}{2}\Lambda M\eta. \end{aligned} \tag{3.4.127}$$

In the limit $\eta \to -\infty$ ($t \to 0$, i.e., on the approach of singularity) they approach the following asymptotic forms (notice that if we assume $p_1 < 0$ and $2p_1 - M < 0$, then $p_1 - M < 0$, provided $M > 0$, which means we have changed expansion of $a(\eta)$ into contraction)

$$\begin{aligned} \alpha(\eta) &\sim -\Lambda(p_1 - M)\eta, \\ \beta(\eta) &\sim \Lambda(p_2 + 2p_1 - M)\eta, \\ \gamma(\eta) &\sim \Lambda(p_3 + 2p_1 - M)\eta, \\ \phi(\eta) &\sim \Lambda M\eta. \end{aligned} \tag{3.4.128}$$

One can check the solutions (3.4.127) by putting them into the constraint (3.4.69) in order to recover the condition (3.4.104), as expected.

Now, we can express the scale factors in terms of the new Kasner parameters

$$\begin{aligned} a &= a_0' t^{p_1'}, \\ b &= b_0' t^{p_2'}, \\ c &= c_0' t^{p_3'}, \\ e^{-\phi} &= d_0' t^{p_4'}, \end{aligned} \tag{3.4.129}$$

where

$$\begin{aligned} p_1' &= -\frac{p_1 - M}{1 + 2p_1 - M}, \\ p_2' &= \frac{p_2 + 2p_1 - M}{1 + 2p_1 - M}, \\ p_3' &= \frac{p_3 + 2p_1 - M}{1 + 2p_1 - M}, \\ p_4' &= \frac{-M}{1 + 2p_1 - M} = -M', \end{aligned} \tag{3.4.130}$$

and

$$\begin{aligned} \Lambda' &= a_0' b_0' c_0' d_0', \\ \eta &= (\Lambda')^{-1} \ln t + \text{const.}, \\ \Lambda' &= (1 + 2p_1 - M)\Lambda. \end{aligned} \tag{3.4.131}$$

If we take the axion field into account ($A \neq 0$) and assume that $a \gg b \gg c$ in (3.4.73)–(3.4.75)), then we obtain

$$\begin{aligned} \alpha_{,\eta\eta} &= \frac{1}{2}\left(A^2 - e^{4\alpha}\right) e^{-2\phi}, \\ \beta_{,\eta\eta} &= \frac{1}{2}\left(A^2 + e^{4\alpha}\right) e^{-2\phi}, \\ \gamma_{,\eta\eta} &= \frac{1}{2}\left(A^2 + e^{4\alpha}\right) e^{-2\phi}, \\ \phi_{,\eta\eta} &= A^2 e^{-2\phi}, \end{aligned} \tag{3.4.132}$$

with $\phi(\eta)$ given by (3.4.99). Notice that, if $M > 0$, then the term $e^{-2\phi}$ increases for $\eta \to -\infty$. If, in turn, $p_1 < 0$, $M > 0$, and $2p_1 - M > 0$, then the term $e^{2(2p_1-M)\eta}$ decreases for $\eta \to -\infty$ and the whole picture is dominated by the axion term $1/2A^2e^{-2\phi}$. It follows that the field equations become isotropic $\alpha_{,\eta\eta} = \beta_{,\eta\eta} = \gamma_{\prime\eta\eta} = 1/2\phi_{,\eta\eta} = 1/2A^2e^{-2\phi}$. The conclusion is that axion isotropizes the model and chaos is impossible in such a case. It seems that the elementary ansatz (see subsection 3.4.1) would allow chaos, but it is not admitted by the BIX geometry. The Kasner solutions in terms of η-time are now given by (compare (3.4.68))

$$\begin{aligned} \alpha(\eta) &= \frac{\Lambda}{2}\left(p_1+M\right)\eta + \text{const.} \equiv \Lambda q_1 \eta + \text{const.}, \\ \beta(\eta) &= \frac{\Lambda}{2}\left(p_2+M\right)\eta + \text{const.} \equiv \Lambda q_1 \eta + \text{const.}, \\ \gamma(\eta) &= \frac{\Lambda}{2}\left(p_3+M\right)\eta + \text{const.} \equiv \Lambda q_1 \eta + \text{const.}, \\ \phi(\eta) &= \Lambda M \eta + \text{const.} \end{aligned} \tag{3.4.133}$$

The solutions which fulfill the above initial conditions (3.4.133) for $\eta \to \infty$ ($p_1 < 0$) are

$$\alpha(\eta) = -\frac{1}{2}\ln\left(\frac{1}{p_1}\cosh p_1\eta\right) + \frac{1}{2}\phi(\eta), \tag{3.4.134}$$

$$\beta(\eta) = \frac{1}{2}\ln\left(\frac{1}{p_1}\cosh p_1\eta\right) + \frac{1}{2}\left(p_1+p_2\right)\eta + \frac{1}{2}\phi(\eta), \tag{3.4.135}$$

$$\gamma(\eta) = \frac{1}{2}\ln\left(\frac{1}{p_1}\cosh p_1\eta\right) + \frac{1}{2}\left(p_1+p_3\right)\eta + \frac{1}{2}\phi(\eta), \tag{3.4.136}$$

$$\phi(\eta) = \ln\left[\cosh M\eta + \sqrt{1-\frac{A^2}{M^2}}\sinh M\eta\right], \tag{3.4.137}$$

or, alternatively

$$\begin{aligned} \alpha(\eta) &= -\frac{1}{2}\ln\left(\frac{1}{2q_1-M}\cosh\left(2q_1-M\right)\eta\right) \\ &+ \frac{1}{2}\phi(\eta), \end{aligned} \tag{3.4.138}$$

$$\beta(\eta) = \frac{1}{2}\ln\left(\frac{1}{2q_1-M}\cosh\left(2q_1-M\right)\eta\right)$$

$$+ \quad (q_1 + q_2 - M)\,\eta + \frac{1}{2}\phi(\eta), \tag{3.4.139}$$

$$\gamma(\eta) = \frac{1}{2}\ln\left(\frac{1}{2q_1 - M}\cosh\left(2q_1 - M\right)\eta\right)$$

$$+ \quad (q_1 + q_3 - M)\,\eta + \frac{1}{2}\phi(\eta), \tag{3.4.140}$$

with $\phi(\eta)$ unchanged.

One can easily check by putting these solutions into the constraint (3.4.69), that the condition (3.4.104) is fulfilled, which in turn ensures that the conditions (3.4.100)–(3.4.101) are fulfilled. In particular, note that, for (3.4.138)–(3.4.140), we need to replace $p_i's$ by $q_i's$. In the limit $\eta \to -\infty$, (that is $t \to 0$), they approach the following forms

$$\begin{aligned} \alpha(\eta) &\sim -\frac{\Lambda}{2}(p_1 - M)\eta, \\ \beta(\eta) &\sim \frac{\Lambda}{2}(p_2 + 2p_1 - M)\eta, \\ \gamma(\eta) &\sim \frac{\Lambda}{2}(p_3 + 2p_1 - M)\eta, \\ \phi(\eta) &\sim -\frac{\Lambda}{2}M\eta, \end{aligned} \tag{3.4.141}$$

or

$$\begin{aligned} \alpha(\eta) &\sim -\Lambda(q_1 - M)\eta, \\ \beta(\eta) &\sim \Lambda(q_2 + 2q_1 - M)\eta, \\ \gamma(\eta) &\sim \Lambda(q_3 + 2p_1 - M)\eta, \\ \phi(\eta) &\sim -\Lambda M\eta. \end{aligned} \tag{3.4.142}$$

Having given the conditions (3.4.100)–(3.4.101), one can express the indices p_2 and p_3 by using p_1 and M, i.e.,

$$p_{2,3} = \frac{1}{2}\left[(M + 1 - p_1) \mp \sqrt{-3p_1^2 + 2p_1\left(M + 1\right) + 1 - M\left(M + 2\right)}\right]. \tag{3.4.143}$$

Since the expression under the square root should be nonnegative, one can extract the restriction (3.4.106) on the permissible values of M. However, we are interested in knowing whether the curvature terms on the right-hand side of the field equations (3.4.73)–(3.4.75) really increase as $\eta \to -\infty$ $(t \to 0)$. This would require either $a^4 e^{-2\phi}, b^4 e^{-2\phi}$, or $c^4 e^{-2\phi}$ to increase if the transition to another Kasner epoch is to occur [114, 123]. Since

$$\begin{aligned} a^4 e^{-2\phi} &\propto t^{(2p_1 - M)} = t^{(1+p_1-p_2-p_3)}, \\ b^4 e^{-2\phi} &\propto t^{(2p_2 - M)} = t^{(1+p_2-p_3-p_1)}, \\ c^4 e^{-2\phi} &\propto t^{(2p_3 - M)} = t^{(1+p_3-p_1-p_2)}, \end{aligned} \tag{3.4.144}$$

we need one of the following three conditions to be fulfilled (remember that we have assumed $p_1 < 0, M > 0$):

$$\begin{aligned} 2p_1 - M &= 1 + p_1 - p_2 - p_3 < 0, \\ 2p_2 - M &= 1 + p_2 - p_3 - p_1 < 0, \\ 2p_3 - M &= 1 + p_3 - p_1 - p_2 < 0. \end{aligned} \tag{3.4.145}$$

The three conditions (3.4.145), with the help of (3.4.143), are equivalent to

$$p_1 < \frac{M}{2}, \tag{3.4.146}$$

$$-3p_1^2 + 2p_1 (M+1) + 1 - M(M+2) > 0. \tag{3.4.147}$$

The last of these conditions, (3.4.147), provides bounds on the possible values of M if a transition is to occur:

$$-2 \leq M \leq \frac{2}{3}. \tag{3.4.148}$$

Now, we see that the regions where the Friedmann isotropic limit is possible (all the Kasner indices equal – this happens for $M = -1 - \sqrt{3}$ and $M = -1 + \sqrt{3}$) are excluded. One can always find the range of the indices for a transition from one Kasner epoch to another to occur in the string frame.

Instead of expressing the conditions for Kasner-type transitions in terms of p_1 and M, we can follow the pattern of [114] and write them in terms of p_1 and p_2. From the conditions (3.4.100)–(3.4.101), we can write

$$p_3 = M + 1 - p_1 - p_2, \tag{3.4.149}$$

$$M = p_1 + p_2 - 1 \pm \sqrt{1 - p_1^2 - p_2^2}, \tag{3.4.150}$$

where the plus sign is for $M > -1, p_3 > 0$ and minus sign for $M < -1, p_3 < 0$. So, p_1 and p_2 must be such that (M real)

$$1 - p_1^2 - p_2^2 \geq 0, \tag{3.4.151}$$

and one of the three conditions (3.4.145) must be fulfilled, i.e., either

$$\begin{aligned} p_1^2 + p_2^2 - p_1 p_2 + p_1 - p_2 &< 0, \\ p_1^2 + p_2^2 - p_1 p_2 - p_1 + p_2 &< 0, \\ p_1^2 + p_2^2 + p_1 p_2 - p_1 - p_2 &> 0, \end{aligned} \tag{3.4.152}$$

for $p_3 > 0$, or

$$\begin{aligned} p_1^2 + p_2^2 - p_1 p_2 + p_1 - p_2 &< 0, \\ p_1^2 + p_2^2 - p_1 p_2 - p_1 + p_2 &< 0, \\ p_1^2 + p_2^2 + p_1 p_2 + p_1 + p_2 &> 0, \end{aligned} \tag{3.4.153}$$

for $p_3 < 0$.

The plot of these conditions is given in Fig. 3.2.

In summary, we have been able to determine the range of values that can be taken by the Kasner indices in the string frame and have proposed a parametrization which describes the evolution of these indices. Also, we have determined the values of the Kasner indices (see Fig. 3.2) for which the spacetime oscillations can really take place. However, now we have to determine whether the oscillations can be stopped once they have started as $t \to 0$. Now let us discuss some relations between the Kasner indices and duality.

3.4.3 Kasner-to-Kasner transitions and duality

The class of homogeneous solutions of the effective-action equations (3.1.18)–(3.1.20) of string cosmology exhibit continuous global $O(d,d)$ symmetry (d is the number of spatial dimensions) which is an example of T-duality within the string theory [166, 168]. It differs from S-duality or the $SL(2,R)$ invariance of superstring models mentioned already (see e.g. [58, 152]). For the class of

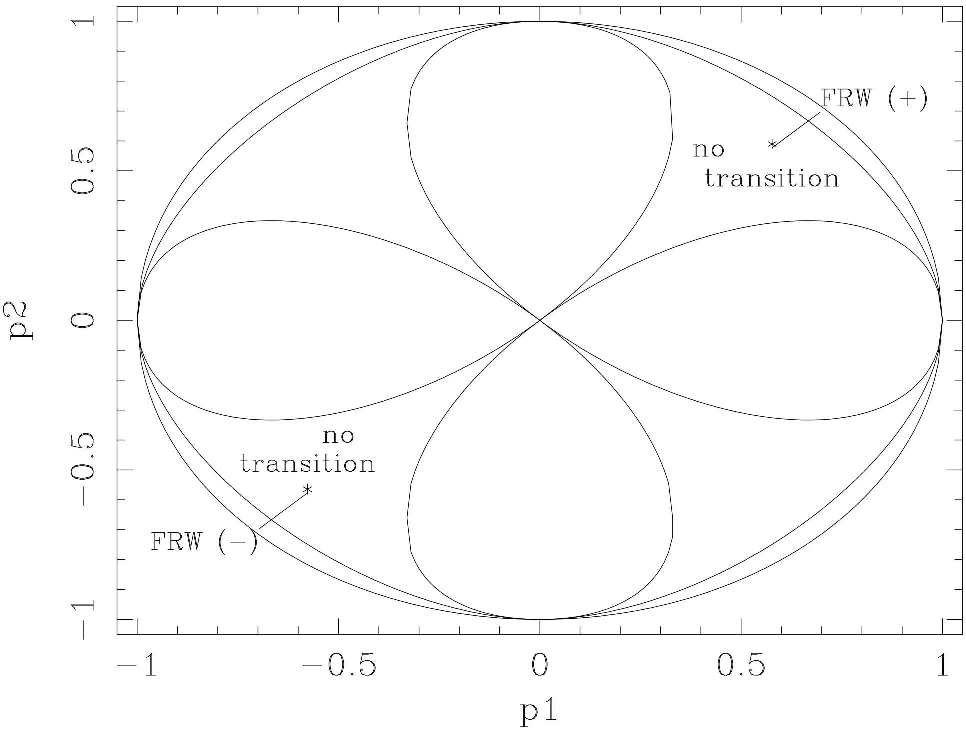

Figure 3.2: The admissible regions for the transitions from one Kasner epoch to another to begin in terms of two Kasner indices p_1 and p_2. The isotropic FRW (+) and (–) points as given by (3.4.152)–(3.4.153) together with two neighbouring regions, are excluded

homogeneous models under consideration T-duality is global $O(3,3)$ invariance under which ($\bar{\phi}$ is the so-called *shifted dilaton* field)

$$\mathbf{M} \to \mathbf{M}' = \mathbf{\Omega}^T \mathbf{M} \mathbf{\Omega}, \qquad \bar{\phi} \equiv \phi - \ln \sqrt{\det \mathbf{G}} \to \bar{\phi}. \tag{3.4.154}$$

Here $\mathbf{\Omega}$ is 6×6 constant matrix satisfying

$$\mathbf{\Omega}^T \mathbf{\Pi} \mathbf{\Omega} = \mathbf{\Pi}, \qquad \mathbf{\Pi} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \tag{3.4.155}$$

and $\mathbf{1}$ is the 3×3 identity matrix and

$$\mathbf{M} \equiv \begin{pmatrix} \mathbf{G}^{-1} & -\mathbf{G}^{-1}\mathbf{B} \\ \mathbf{B}\mathbf{G}^{-1} & \mathbf{G} - \mathbf{B}\mathbf{G}^{-1}\mathbf{B} \end{pmatrix}, \tag{3.4.156}$$

where $\mathbf{G} = (g_{ij})$ and $\mathbf{B} = (B_{ij})$ are 3×3 matrices. Any 6×6 constant matrix $\mathbf{\Omega}$ obeying (3.4.155) generates new solutions $\mathbf{M}'$ from the original set $\mathbf{M}$. Notice that for full $O(3,3)$ symmetry both $\mathbf{G}$ and $\mathbf{B}$ have to be functions of time. This is especially important for the antisymmetric tensor potential B_{ij} which is time-dependent and so leads to a space-dependent pseudoscalar axion field, h (elementary ansatz). As we proved already in subsection 3.4.1, the elementary ansatz is not compatible with (even axisymmetric) Bianchi type IX geometry. Thus, because of our solitonic ansatz (3.4.21), the full $O(d,d)$ symmetry is broken and it can only be recovered if the antisymmetric tensor field B_{ij} (or axion h) vanishes. If this happens, we can make a choice $\mathbf{\Omega} = \mathbf{\Pi}$ and consider the "scale factor duality" where

$$\begin{aligned} a'^2 &\rightarrow \frac{1}{a^2}, \\ b'^2 &\rightarrow \frac{1}{b^2}, \\ c'^2 &\rightarrow \frac{1}{c^2}, \\ \phi' &\rightarrow \phi - 2\ln(abc). \end{aligned} \tag{3.4.157}$$

It is useful to define the logarithm of an average scale factor $\bar{\beta}$ and the so-called shifted dilaton $\bar{\phi}$ defined by [82, 102]

$$\begin{aligned} \bar{\beta}_i &= \frac{1}{\sqrt{3}} \ln a_i, \\ \bar{\beta} &= \frac{1}{\sqrt{3}} \ln(abc), \\ \bar{\phi} &= \phi - \sqrt{3}\bar{\beta}. \end{aligned} \tag{3.4.158}$$

Using (3.4.105) and (3.4.126) we have the relations (3.4.158) in terms of Kasner indices, i.e.,

$$\begin{aligned} \bar{\phi} &= \phi - \ln\Lambda - (M+1)\ln t, \\ \bar{\beta}_i &= \frac{1}{\sqrt{3}} (\ln a_{0i} + p_i \ln t), \\ \bar{\beta} &= \frac{1}{\sqrt{3}} [\ln\Lambda + (M+1)\ln t], \end{aligned} \tag{3.4.159}$$

where $a_{0i} = \{a_0, b_0, c_0\}$. In these variables the duality symmetry is just expressed by

$$\begin{aligned} \bar{\beta}_i(t) &= -\bar{\beta}_i(t), \\ \bar{\beta}(t) &= -\bar{\beta}(t), \\ \bar{\phi}(t) &= \bar{\phi}(t); \end{aligned} \tag{3.4.160}$$

or, in terms of Kasner indices, it reads $p_i \to -p_i, M+1 \to -(M+1)$. After inclusion of time symmetry we have [102]

$$\begin{aligned} \bar{\beta}_i(t) &= -\bar{\beta}_i(-t), \\ \bar{\beta}(t) &= -\bar{\beta}(-t), \\ \bar{\phi}(t) &= \bar{\phi}(-t). \end{aligned} \tag{3.4.161}$$

In the isotropic case $p_i = \pm 1/\sqrt{3}$ and we recover exactly the case given in [102].

The same relations can be written down using the time coordinate η instead of t. Using the exact expressions, (3.4.126), for Kasner solutions we have

$$\begin{aligned} \bar{\beta} &= \frac{\Lambda}{\sqrt{3}}(M+1)\eta, \\ \bar{\phi} &= \phi - \sqrt{3}\bar{\beta} - \Lambda\eta, \\ \bar{\beta}_i &= \frac{\Lambda}{\sqrt{3}}p_i\eta. \end{aligned} \tag{3.4.162}$$

So, one could relate these by duality symmetries

$$\bar{\phi}(\eta) \to \bar{\phi}(-\eta), \tag{3.4.163}$$

$$\bar{\beta}(\eta) \to -\bar{\beta}(-\eta), \tag{3.4.164}$$

for $\eta \to \pm\infty$ respectively, using chaotic changes $p_i \to -p_i$.

In our analysis we have used the standard procedure of assuming that the mixmaster model is well described by a sequence of Kasner-to-Kasner transitions. This assumption is an analogue of that of steep walls in the Hamiltonian approach. Numerical studies of mixmaster models show it to be a good approximation even in the presence of chaotic behaviour. We do not find chaotic behaviour and so the approximation should be better over long periods of evolution. We note also that the approximations made ($a \gg b \gg c$) to study single Kasner-to-Kasner transitions reduce equations to these of the axisymmetric case. This describes a single Kasner-to-Kasner transition. We therefore expect the duality relationships characterizing Kasner-to-Kasner transitions to provide good approximations to the properties of the exact mixmaster behaviour and we do not see any reason to consider non-Abelian dualities of this exact model [93]. We do not know whether the string Bianchi type IX model is integrable in general.

3.4.4 Hamiltonian approach to Bianchi IX String Cosmologies

In this subsection we formulate a generalized Kasner model in Hamiltonian formalism as in [89, 114, 186] in order to discuss the conditions for an infinite sequence of scatterings to occur against the walls of the curvature potential. As in the previous sections we discuss the problem in both the Einstein and the string frames. We also introduce the axion frame [55], in which axion is minimally coupled.

Einstein Frame

We introduce the following standard parametrization for the Einstein-frame scale factors

$$\begin{aligned} \tilde{a} &= e^{\tilde{\alpha}+\psi_{+}+\sqrt{3}\psi_{-}}, \\ \tilde{b} &= e^{\tilde{\alpha}+\psi_{+}-\sqrt{3}\psi_{-}}, \\ \tilde{c} &= e^{\tilde{\alpha}-2\psi_{-}}, \end{aligned} \tag{3.4.165}$$

and we define the potential, which describes the spatial curvature anisotropy (3.4.13) felt by scale factors in the Einstein-frame by

$$\tilde{V}(\psi_{\pm}) = e^{-2\tilde{\alpha}}V(\psi_{\pm}), \tag{3.4.166}$$

where

$$V(\psi_{\pm}) = \frac{1}{2}\left[e^{-8\psi_{+}} + 2e^{4\psi_{+}}\left(\cosh 4\sqrt{3}\psi_{-} - 1\right) - 4e^{-2\psi_{+}}\cosh 2\sqrt{3}\psi_{-}\right]. \tag{3.4.167}$$

Using (3.4.165)–(3.4.166), the Eqs. (3.4.57)–(3.4.60) read as

$$\tilde{\alpha}'^2 = \psi_+'^2 + \psi_-'^2 + \frac{1}{12}\phi'^2 + \frac{1}{12}A^2 e^{-2\phi-6\tilde{\alpha}} + \frac{1}{6}e^{-2\tilde{\alpha}}V(\psi_\pm). \tag{3.4.168}$$

This is the Hamiltonian constraint. The Einstein-frame action in terms of the scale factors (3.4.165) after integrating out spatial variables is given by

$$S = \int d\tilde{t} e^{3\tilde{\alpha}} \left[-6\tilde{\alpha}'^2 + 6\psi_+'^2 + 6\psi_-'^2 + \frac{1}{2}\phi'^2 + \frac{1}{2}A^2 e^{2\phi}\sigma'^2 + e^{-2\tilde{\alpha}}V(\psi_\pm) \right], \tag{3.4.169}$$

and the conjugate momenta are

$$\begin{aligned} \pi_{\tilde{\alpha}} &= -12\tilde{\alpha}' e^{3\tilde{\alpha}}, \\ \pi_+ &= 12\psi_+' e^{3\tilde{\alpha}}, \\ \pi_- &= 12\psi_-' e^{3\tilde{\alpha}}, \\ \pi_\phi &= \phi' e^{3\tilde{\alpha}}, \\ \pi_\sigma &= \sigma' e^{2\phi+3\tilde{\alpha}} = \text{const.} = \text{A}, \end{aligned} \tag{3.4.170}$$

so the Hamiltonian is

$$H = -\frac{\pi_\alpha^2}{24} + \frac{\pi_+^2}{24} + \frac{\pi_-^2}{24} + \frac{\pi_\phi^2}{2} + \frac{\pi_\sigma^2}{2} + e^{4\tilde{\alpha}}V(\psi_\pm). \tag{3.4.171}$$

Now, we follow the standard discussion of the potential walls

$$\bar{V} = e^{4\tilde{\alpha}}V(\psi_\pm) \tag{3.4.172}$$

being hit by a particle moving in the potential well [89]. In the region $\psi_+ \ll -1$ and $\psi_- \approx 0$, the approximate distance from the origin of coordinates ψ_+ and ψ_- to the wall is given by

$$D = -\frac{1}{2}\tilde{\alpha}, \tag{3.4.173}$$

while the maximum apparent velocity of this wall is

$$v_{max} = \tilde{\alpha}'. \tag{3.4.174}$$

The velocity of a particle moving against the walls is

$$v_p = \sqrt{\psi_+'^2 + \psi_-'^2}, \tag{3.4.175}$$

and it will not be scattered infinitely many times if there is some region of the potential which the particle enters and from which it cannot catch up with the wall, i.e., if

$$\begin{aligned} v_p &= \sqrt{\psi_+'^2 + \psi_-'^2} < \tilde{\alpha}' \\ &\approx \sqrt{\psi_+'^2 + \psi_-'^2 + (1/12)\phi'^2 + (1/12)A^2 e^{-2\phi - 6\tilde{\alpha}}}. \end{aligned} \qquad (3.4.176)$$

Clearly, this condition is fulfilled in every case unless $\phi' = A = 0$ (no dilaton and axion – that is, the general relativity vacuum regime), which reflects the fact that a particle cannot be scattered infinitely many times and that *there is no chaos in the Einstein frame.* This result is expected since in the Einstein frame both dilaton and axion fields behave as stiff fluids with the equation of state $p = \varrho$.

String Frame and Axion frame

In the string frame we can use the same parametrization as in (3.4.165), but we just drop the tildes. The potential (3.4.166) can also be used without tildes. In that parametrization (3.4.55) becomes

$$\dot{\alpha}^2 = \dot{\psi}_+^2 + \dot{\psi}_-^2 - \frac{1}{6}\dot{\phi}^2 + \frac{1}{6}e^{-2\alpha}V(\psi_\pm) + \frac{1}{12}A^2 e^{-6\alpha} + \dot{\alpha}\dot{\phi}. \qquad (3.4.177)$$

After applying the variables $\bar{\beta}$ and $\bar{\phi}$ defined by (4.5) we can remove the $\dot{\phi}\dot{\alpha}$ term, obtaining

$$\dot{\bar{\phi}}^2 = \dot{\bar{\beta}}^2 + 6\dot{\psi}_+^2 + 6\dot{\psi}_-^2 + e^{-2\frac{\bar{\beta}}{\sqrt{3}}}V(\psi_\pm) + \frac{1}{2}A^2 e^{-2\sqrt{3}\bar{\beta}}. \qquad (3.4.178)$$

Following the analysis given in [102], we apply a new time coordinate τ defined as

$$dt = d\tau e^{-\bar{\phi}}, \qquad (3.4.179)$$

and define a new variable, y, which is the logarithm of an averaged scale factor in the conformally related axion frame [55]. This is given by

$$y \equiv \sqrt{3}\bar{\phi} + \bar{\beta} = \frac{1}{\sqrt{3}}\ln(abc), \qquad (3.4.180)$$

and brings (3.4.178) to the form $((...)_\tau = d(...)/d\tau)$

$$y_\tau^2 = \phi_\tau^2 + 12\psi_{+\tau}^2 + 12\psi_{-\tau}^2 + A^2 e^{-2\phi} + 2e^{-\frac{2}{\sqrt{3}}y} V(\psi_\pm). \tag{3.4.181}$$

Eq. (3.4.181) is, in fact, the Hamiltonian constraint obtained from the action $(\lambda_s^2 = 8\pi G)$

$$S = \frac{\lambda_s}{4} \int d\tau \left[\phi_\tau^2 - y_\tau^2 + 12\left(\psi_{+\tau}^2 + \psi_{-\tau}^2\right) - A^2 e^{-2\phi} - 2e^{-\frac{2}{\sqrt{3}}y} V(\psi_\pm)\right]. \tag{3.4.182}$$

The canonical momenta are then

$$\begin{aligned} \pi_\phi &= \frac{\lambda_s}{2}\phi_\tau, \\ \pi_y &= -\frac{\lambda_s}{2} y_\tau, \\ \pi_+ &= 6\lambda_s \psi_{+\tau}, \\ \pi_- &= 6\lambda_s \psi_{-\tau}, \end{aligned} \tag{3.4.183}$$

and the Hamiltonian is just

$$H = \frac{1}{\lambda_s}\left[\pi_\phi^2 - \pi_y^2 + \frac{1}{12}\left(\pi_+^2 + \pi_-^2\right) + \frac{\lambda_s^2}{4} A^2 e^{-2\phi} + \frac{\lambda_s^2}{2} e^{-\frac{2}{\sqrt{3}}y} V(\psi^\pm)\right]. \tag{3.4.184}$$

Following the analysis of the Einstein frame, we see that the maximum apparent velocity of the wall at $\psi_+ \ll -1, \psi_- \approx 0$ is given by

$$v_{max} = \frac{1}{2\sqrt{3}} y_\tau, \tag{3.4.185}$$

and the condition for chaotic scatterings to cease is just that

$$v_p = \sqrt{\psi_{+\tau}^2 + \psi_{-\tau}^2} < \frac{1}{2\sqrt{3}} y_\tau \approx \sqrt{\psi_{+\tau}^2 + \psi_{-\tau}^2 + \frac{1}{12}\phi_\tau^2 + \frac{1}{12}A^2 e^{-2\phi}}. \tag{3.4.186}$$

This is clearly fulfilled except in the general relativity case where $\phi = A = 0$ (i.e., no axion and dilaton fields). This gives our final conclusion that *there is no chaos in BIX string cosmology in the string or axion frames.*

It is not surprising that the physical behaviour should be similar in every frame [40, 83, 201], so that if there is no chaos in the Einstein frame there should not be chaos in any other frame. String theory appears to impose too much symmetry through its duality invariances for chaos to appear.

To conclude, we carried out a detailed analysis of the spatially homogeneous universes of Bianchi type IX in the context of the common sector of effective superstring actions. These universes are of special interest in relativistic cosmology because they display chaotic behaviour in vacuum and in the presence of fluids with $p < \rho$. They were originally termed 'Mixmaster' universes by Misner because they offered the possibility for light to travel all the way around the universe in different directions. Moreover, they are the most general closed universes which are spatially homogeneous and may be closely related to parts of the general solution of Einstein's equations in the neighborhood of a strong curvature singularity of the sort that characterizes the initial state of general relativistic cosmological models. This behaviour has also been extensively investigated because it is of intrinsic mathematical interest. It is also known that its occurrence in general relativity depends upon the dimensionality of space. We investigated the string cosmological equations for the type IX metric. We found that chaotic behaviour does not occur in string cosmology in either the Einstein or the string or the axion frames. While it is possible for finite sequences of oscillations to occur in the scale factors' evolution on approach to $t = 0$, these oscillations cannot continue indefinitely. They inevitably terminate in a state in which all the three orthogonal scale factors decrease with decreasing time monotonically on approach to the initial singularity. We investigated the detailed sequences of evolutionary changes that can take place in the evolution during the finite sequences of oscillations between epochs which are well approximated by Kasner universes. We found that the duality symmetry required of the string evolution introduced new invariances for the possible changes in the Kasner parameters in addition to those which characterise the Kasner-to-Kasner cycles of oscillations. The requirements of duality invariance on the evolution of the metric appear to be so constraining that chaotic behaviour is excluded. We have obtained these results in two complementary ways: by direct matching of asymptotic expansion to the solutions of the system of non-linear ordinary differential equations of string cosmology and by use of the Hamiltonian formulation of cosmology. In the Hamiltonian picture the evolution of the type IX string cosmology is represented as the motion of a 'universe point' inside a potential that is open only along three narrow channels. The walls of this 'almost closed' potential of Hénon-Heiles type expand outwards as the singularity is reached. Whereas in the vacuum models of general relativity, the universe point always catches the walls and bounces chaotically around within the potential, in string theory the universe point need never catch the walls. If it is moving towards a wall at a very oblique angle then the normal component of its velocity towards the wall can become too small for it ever to catch the wall. In general, we find that this situation always arises after a finite number of collisions have occurred in the Mixmaster string cosmology. The resulting asymptotic state is therefore similar to that in a model with no potential walls at all; that is the Bianchi type I or Kasner universe.

3.5 Gödel string cosmologies

3.5.1 Stringy Gödel universes

The Gödel metric describes space and time homogeneous spacetime [107] with 5-parameter group of symmetry. Its most amazing feature is that it allows for existence of closed timelike curves (CTCs) in such homogeneous spacetime [118]. In this section we consider Gödel spacetime within the framework of string cosmology with the line element in cylindrical coordinates (t, r, z, ψ) given by either of the two forms [182]

$$ds^2 = -\left[dt + C(r)d\psi\right]^2 + D^2(r)d\psi^2 + dr^2 + dz^2, \tag{3.5.1}$$

or

$$ds^2 = -dt^2 - 2C(r)dtd\psi + G(r)d\psi^2 + dr^2 + dz^2,$$

where the radial functions have the form

$$C(r) = \frac{4\Omega}{m^2}\sinh^2\left(\frac{mr}{2}\right), \tag{3.5.2}$$

$$D(r) = \frac{1}{m^2}\sinh(mr), \tag{3.5.3}$$

$$G(r) = \frac{4}{m^2}\sinh^2\left(\frac{mr}{2}\right)\left[1+\left(1-\frac{4\Omega^2}{m^2}\right)\sinh^2\left(\frac{mr}{2}\right)\right], \tag{3.5.4}$$

with m and Ω constants. With metric (3.5.1) one is able to *avoid* the existence of CTCs provided

$$G(r) = D^2(r) - C^2(r) > 0. \tag{3.5.5}$$

This is consistent with the 'chronology protection conjecture' of Hawking [116]. In a Gödel universe, the four-velocity of matter is $u^\alpha = \delta_0^\alpha$ and the rotation vector is $V^\alpha = \Omega\delta_3^\alpha$ while the vorticity scalar is given by $\omega = \Omega/\sqrt{2}$. The original Gödel metric of general relativity has $m^2 = 2\Omega^2$ and obviously contradicts (3.5.5) [107]. There has been extensive discussion of the generality and significance of the presence of CTCs in the Gödel metric in general relativity [118, 173].

The only nonvanishing components of the Riemann tensor in an orthonormal frame permitted by the spacetime homogeneity of the Gödel universe are constant, [2], with

$$R_{0101} = R_{0202} = \frac{1}{4}\left(\frac{C'}{D}\right)^2 = \Omega^2, \qquad R_{1212} = \frac{3}{4}\left(\frac{C'}{D}\right)^2 - \frac{D''}{D} = 3\Omega^2 - m^2. \tag{3.5.6}$$

Here the prime means the derivative with respect to r. For Riemann tensor given by (3.5.6) the Gauss-Bonnet term $R^2_{GB} \equiv R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ vanishes. This term appears in redefined $O(\alpha')$ actions (3.1.11) (see e.g. [85, 166]). Because of the metric symmetry we assume that the dilaton depends only on the coordinate along the axis of rotation, z, so

$$\phi = \phi(z) = fz + \phi_0, \tag{3.5.7}$$

where f and ϕ_0 are constants. For the axion, a short analysis similar to that given in [16, 20, 21] shows that the only possible ansatz which is consistent with the form (3.5.7) for the dilaton is

$$H_{012} = -H^{012} = E, \tag{3.5.8}$$

with E constant. This can also be expressed in terms of the pseudoscalar axion field, h, [16, 20, 57], by

$$h(z) = \frac{E}{f} \exp\left(-fz - \phi_0\right) + h_0. \tag{3.5.9}$$

The ansätze (3.5.7)–(3.5.8) guarantee that the axion's equation of motion is satisfied to zeroth order in α'. Since the axion field's 3-form strength is defined by an external derivative of the antisymmetric tensor potential, $B_{\mu\nu}$, one can easily show that the only nonvanishing component of the potential is time-dependent [18] and this is another example of an elementary ansatz for axion field,i.e.,

$$B_{12}(t) = Et + t_0. \tag{3.5.10}$$

This is expected from the discussion of [16, 20, 57] since this restriction occurs in every case where the spacetime possesses a distinguished direction. It should then possess $O(n-1, n-1)$ symmetry (despite the fact the metric is not time-dependent): which is an example of T-duality.

Note the generality of the action (3.1.11): it is given for any spacetime dimension n, and it contains the full spectrum of graviton, axion, and dilaton. It also possesses a general $O(n-1, n-1)$ symmetry [166]. Since this manifests itself in complicated forms in individual solutions, we first discuss two special cases before giving the general Gödel solution for this string theory.

3.5.2 Zeroth-order in α' universes

The field equations (3.1.12)–(3.1.13) (which are in the string frame) to zero-order in α', together with (3.5.6)–(3.5.8) with $D = 4$, possess a Gödel solution with metric (3.5.1)–(3.5.4) if the following relations hold between the constants

Ω, m, E, f and Λ:

$$\Omega^2 = \frac{m^2}{4} = \frac{E^2}{4} = \frac{-2\Lambda - f^2}{4}. \tag{3.5.11}$$

When $\Lambda = 0$ the relation (3.5.11) has no meaning for $m^2 > 0$, but it can be applied for $m^2 \equiv -\mu^2 < 0$, in which case there exists an infinite sequence of both causal and acausal regions (see [182]). We do not consider such solutions here.

From the relations (3.5.11) one can easily deduce that the axion field must not vanish if a solution is to exist and act as a source of rotation. This is consistent with the known repulsive behaviour of the axion acting as a torsion field, which leads to bouncing solutions within the string theory [28]. However, from (3.5.11), there is another constraint,

$$\Lambda < -\frac{f^2}{2}, \tag{3.5.12}$$

which shows that the cosmological term has to be negative. Formally, the dilaton can vanish without disrupting the causal structure of the solution (3.5.1), but in that case the axion plays the role of a scalar field minimally coupled to gravity – the case studied in [182].

These results are quite different from the situation in general relativity with both a scalar field and electromagnetic field present [182]. It is known [57] that the action (3.1.11) can be transformed to the Einstein frame where the field equations resemble Einstein gravity with source terms given by axion and dilaton fields [57]. Using the ansätze (3.5.7)–(3.5.9) one gets relations (compare [182])

$$\Omega^2 = \frac{m^2}{4} = \frac{f^2}{4} = -2\Lambda. \tag{3.5.13}$$

We see also that (in analogy to the electromagnetic field that is present there), we cannot admit the axion and obtain a causal model (3.5.1) in the Einstein frame because there is no way to fulfil the dilaton equation of motion. This might be an artifact of choosing Gödel geometry as a model of the universe and may disappear once one considers other background geometries [45]. Note that

Λ has to be negative in both frames. For $\lambda_0 = 0$ in (3.1.11), these conclusions also hold for superstrings.

3.5.3 First order in α', no axion universes

Now we add the α' terms to the equations (3.1.12)–(3.1.13), but neglect the axion. The resulting equations are reduced to three polynomial constraints,

$$2\Omega^2 - 2\alpha'\Omega^4 = 0, \tag{3.5.14}$$

$$2\Omega^2 - m^2 + \alpha'\left(10\Omega^4 - 6\Omega^2 m^2 + m^4\right) = 0, \tag{3.5.15}$$

$$2\Omega^2 - 2m^2 - 2\Lambda - f^2 + \alpha'\left(11\Omega^4 - 6\Omega^2 m^2 + m^4\right) = 0. \tag{3.5.16}$$

We have three equations and five constants and so two (say, Λ and f) can be chosen arbitrarily.

From the equation (3.5.14) we immediately have the simple relation

$$\alpha' = \frac{1}{\Omega^2}, \tag{3.5.17}$$

which gives the velocity of rotation of the Gödel universe in terms of the inverse string tension. This relation gives a simple connection between micro and macrophysics in this spacetime, with a balance between string tension and rotation. Using (3.5.17), subtracting (3.5.15) and (3.5.16), we find,

$$m^2 + 2\Lambda + f^2 = \Omega^2. \tag{3.5.18}$$

From (3.5.15) and (3.5.17) we calculate the possible values of Ω in terms of m to be,

$$\Omega_+^2 = \frac{1}{3}m^2, \tag{3.5.19}$$

or

$$\Omega_+^2 = \frac{1}{4}m^2. \tag{3.5.20}$$

The case (3.5.19) allows CTCs ($G(r) < 0$ in (3.5.5), just as in general relativity. For the case (3.5.20), after using (3.5.16), we obtain the relations

$$\Omega^2 = \frac{1}{\alpha'} = \frac{m^2}{4} = \frac{-2\Lambda - f^2}{3}, \tag{3.5.21}$$

and the condition (3.5.12) must still hold. This can also be related to the number of dimensions $D = 4$, using (3.1.17) to remove Λ from the relation (3.5.21), so

$$\Omega^2 = \frac{1}{\alpha'} = \frac{m^2}{4} = \frac{3}{35}f^2, \tag{3.5.22}$$

and the cosmological constant (3.1.17) must be negative, in agreement with [2].

3.5.4 General first order in α' universes

In the case of the bosonic string ($\lambda_0 = -1/4$), using (3.1.15)–(3.1.17), the field equations (3.1.12)–(3.1.13) for the Gödel metric reduce to the three polynomials,

$$2\Omega^2 - \frac{1}{2}E^2 + \frac{1}{2}\alpha'\left[-4\Omega^4 + 4\Omega^2 E^2 + \frac{5}{4}E^4\right] = 0, \tag{3.5.23}$$

$$2\Omega^2 - m^2 + \frac{1}{2}E^2$$
$$+ \frac{1}{2}\alpha'\left[20\Omega^4 - 12\Omega^2 m^2 + 2m^4 - 2E^2\left(m^2 - 2\Omega^2\right) - \frac{5}{4}E^4\right] = 0, \tag{3.5.24}$$

$$2\Omega^2 - 2m^2 - 2\Lambda - f^2 + \frac{1}{2}E^2$$
$$+ \frac{1}{4}\alpha'\left[44\Omega^4 - 24\Omega^2 m^2 + 4m^4 - 2E^2\left(m^2 - \Omega^2\right) - \frac{5}{4}E^4\right] = 0. \tag{3.5.25}$$

The axion equation of motion (3.1.14) is also fulfilled in this general case. We now have three equations and six constants, leaving three (say, Λ, f and E) arbitrary.

From (3.5.23) and (3.5.24) we obtain

$$\Omega^2 = \frac{m^2}{4}. \tag{3.5.26}$$

This confirms that it is possible to obtain a Gödel solution with no CTCs which fulfils (3.5.5) in the general case. The value of α' can now be expressed in terms of the velocity of rotation of the universe, Ω, and the strength of axion field, E, from (3.5.23), which gives,

$$\alpha' = \frac{4\Omega^2 - E^2}{(4\Omega^2 + E^2)\left(\Omega^2 E - \frac{5}{4}E^2\right)}. \tag{3.5.27}$$

The relation between other constants can be obtained from (3.5.25). Similar calculations can also be performed for heterotic strings (with $\lambda_0 = -\frac{1}{8}$ in (3.1.11)) which demonstrate the existence of Gödel solutions without CTCs within that theory too. The superstring case ($\lambda_0 = 0$) does not generate a solution of this type to this order in α' because the quadratic correction term vanishes in the action (3.1.11) and the dominating term in the action is quartic in the Riemann curvature tensor [111].

To summarize, we have found a class of Gödel universes without closed timelike curves within the framework of low-energy-effective string theories. First, to zero-order in the inverse string tension α', we investigated the situation in both the Einstein frame (as studied already in general relativity in [182]) and in the string frame. We found that the axion cannot be introduced in the Einstein frame but plays a crucial role in the string frame, where it cannot be neglected. Then we extended the analysis to include the full $O(\alpha')$ action with both dilaton and axion taken into account. By including terms of the first order in α' in the field equations, we found that Gödel universes *without* closed timelike curves are also possible in the general case. Our solutions display a simple relation between the inverse string tension parameter α' and the velocity of rotation of the universe (3.5.17) which provides a direct link between micro and macrophysics. This is the first class of exact solutions that has been found for a string theory where terms of first order in α' are admitted in the field equations (a method for generating exact solutions in Bianchi type I universes was given by Mueller [172]). All the Gödel universes we have found require negative cosmological constant which, in general, can be related to the number of spacetime dimensions if multidimensional cases are also investigated. Our results, obtained for bosonic strings, are also valid for heterotic and superstrings, as is evident from the form of the action, although the numerical values of the constants will change with the value of the parameter λ_0. The simplicity of the final forms of the solutions we have found for Gödel universes suggests that the symmetries of string theory have the power to exclude unwanted peculiarities in the causal structure of spacetime that are permitted by general relativity. They also suggest a path towards finding further exact solutions at higher order in the string tension parameter. Although the Gödel universe does not describe our universe, it does isolate a general feature that rotation can introduce into relativistic spacetimes. In this sense it is physically relevant. Historically, the Gödel universe revealed the possibility of time travel in general relativity. Our analysis does not prove that there cannot be time travel in string theory. However, its absence from Gödel metrics at the order we have examined should act as a stimulus to further investigations at the higher orders.

Chapter 4

“Graceful-exit” problem

4.1 “Exit-out” of superinflation

The admission of negative time branch in the solutions of superstring cosmology (see Section 3.2) justifies its name as *pre-big-bang* cosmology. In fact, this name can be a bit misleading and it is useful to explain what we mean by that. It seems that the main point is about boundary conditions. In standard picture, which is expressed in terms of Hartle-Hawking or Vilenkin boundary conditions [115, 202], the universe emerges quantum mechanically (big-bang) into the de Sitter phase with microscopic radius a_0 (of large curvature), then undergoes inflation, reheating and radiation-dominated expansion (see Fig. 4.1).

In the pre-big-bang cosmology the universe begins with a trivial perturbative string Minkowski vacuum [40, 201], i.e., the phase of weak coupling $g_s \ll 1$ and small curvature. Then, it undergoes superinflation which is a power-law inflation, eventually reaching ‘stringy’ phase (strong curvature and coupling regime), where the effective action (3.1.9) breaks down. It finally evolves towards a radiation-dominated expanding phase. In pre-big-bang scenario the initial conditions are, in fact, moved back before the point which we used to call big-bang in standard approach. In order to be precise what we mean by pre-big-bang, it is useful to interpret the whole scenario as a change of state for the universe in some analogy to what happens in β-decay, where proton, electron and antineutrino are created out of neutron (which no longer exists)

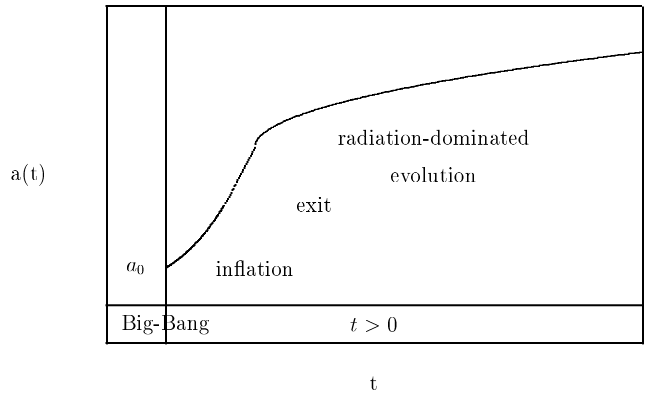

Figure 4.1: Standard big-bang cosmology. The universe appears quantum-mechanically as de Sitter universe with microscopic size a_0 (high curvature) and undergoes inflationary phase driven by the potential energy of the inflaton. Then, it exits inflation, eventually approaching standard radiation-dominated evolution. Big-bang is at the moment t = 0 and the whole picture involves only positive times (t ¿ 0)

in a kind of a 'big-bang' for them [94]. An initial state is a phase **A** ($t \to -\infty$) which is the perturbative string Minkowski vacuum, then there is the phase **B** ($t = 0$) of strong coupling and curvature and finally one has state **C** ($t > 0$), which is the radiation-dominated evolution (cf. Fig. 4.1). The main problem of this picture refers to phase **B** since once superinflationary universe has approached this phase, the action (3.1.9) is no longer valid (it is valid for $g_s \ll 1$ – cf. (1.1.3) – or, alternatively for $\mid t \mid > \lambda_s \propto l_{pl}$ where l_{pl} is the Planck length [65, 126, 192]). A possible transition through the phase **B** is commonly referred to as "graceful exit" problem of pre-big-bang cosmology. It has been proven as a new type of "no-go" theorem [36, 37, 39, 127, 128] that there is no way to connect classically the duality-related solutions and to overcome the "graceful exit problem" in the simplest models of string cosmology. With respect to this result, it seems that the classical scenario breaks down and that one needs to take quantum effects into account to avoid the singularity. It is widely believed that the singularity of strong coupling and curvature can be regularized once the full string theory is applied. First of all, the action (3.1.9) is truncated to the lowest order in two expansions (cf. Chapter 3). As for the α' expansion its more general form than 3.1.9 (first in α' order as given by (3.1.11) or higher orders) can be used to possibly avoid singularity.

The second type of expansion is of quantum nature (cf. Chapter 3) and this is the 'string-loop'-expansion in string coupling constant (1.1.3) [32, 43]. The simplest one-loop corrected action reads as

$$S = S_{tree} - \alpha'\omega \int d^4x\sqrt{-g}\zeta\sigma F[R^2, (\partial_\mu\phi)^4, R(\partial_\mu\phi)^2, \ldots], \tag{4.1.1}$$

where $\omega =$ const. of the order of 1, $\zeta(\sigma) \sim -(2\pi/3)\cosh\sigma$ with σ being the modulus field parametrizing compactified dimensions.

As for the actions of the type (3.1.11) (α'-expansion), it is only possible to regularize branches **1** and **3** and still higher corrections are neccessary. As for the action (4.1.1) it is possible to regularize curvature but not dilaton and also branches **1** and **4** do not interchange (dilaton remains in pre-big-bang phase). Some possible progress in this field was also made in [4].

An alternative, though somehow a compromise with standard approach to the 'graceful-exit' problem is to apply Wheeler-de-Witt equation and consider quantum mechanical tunneling through the 'stringy' phase of (presumably finite) strong curvature [102]. The probability amplitude of tunneling from pre-big-bang branch **1** into the post-big-bang branch **4** has been calculated and it is a function of the potential term (cosmological constant) applied. The amplitude has similar functional dependence as the amplitude of tunneling in Vilenkin boundary conditions approach. This approach is be the main objective of this Chapter.

Now, let us strongly emphasize that all the above effects (superinflation etc.) are the properties of the theory in the string frame. In fact, in the conformally related Einstein frame (cf. 3.1.21) where the action is given by (3.1.22) the isotropic Friedmann models behave in a similar way as stiff-fluid models $p = \varrho$ (pressure = energy density) of general relativity, and superinflationary phase in string frame corresponds to a collapse of the universe in the Einstein frame (both for $t < 0$). Then, really nothing special happens in the Einstein frame. However, it is believed that, although in the weak coupling limit both frames are equivalent, this is not the case when strong coupling limit is approached, where all the physical (or 'stringy') effects should be referred to the string frame.

From now on we use the common sector effective action (3.1.9), and, in the next sections we apply the formalism of canonical quantum gravity to describe a quantum transition form the "pre-big-bang" phase to the "post-big-bang" phase through the singularity [97, 102]. More precisely, in the minisuperspace comprising scale factor a and dilaton ϕ, a solution to the Wheeler-DeWitt equation can be found after imposing boundary conditions in the strong coupling regime $\phi \to \infty$. This solution was interpreted as describing a reflection in minisuperspace through the singularity. We study this problem for various minisuperspace models – from these based on isotropic geometries, to those based on homogeneous geometries of Bianchi types and Kantowski-Sachs.

4.2 Quantum string minisuperspaces

The quantum string cosmology that we consider in Sections 4.2–4.5 is based on the quantized tree-level string effective action (3.1.9) [102, 148, 149]. The quantized actions of this type with minimally or conformally coupled scalar fields were also considered in [175].

We will study string minisuperspace models for cosmologies where the dilaton and metric are homogeneous on spatial hypersurfaces. For spatially homogeneous Bianchi models we use the parametrization of Misner, Thorne and Wheeler [171] with the 3-metric given by

$$g_{ij} = e^{2\alpha}\left(e^{2\psi}\right)_{ij}, \tag{4.2.1}$$

with the scale factor e^{α} and the matrix

$$(\psi_{ij}) \equiv \mathrm{diag}(\psi_+ + \sqrt{3}\psi_-, \psi_+ - \sqrt{3}\psi_-, -2\psi_+), \tag{4.2.2}$$

which defines the anisotropy parameters $\psi_\pm$. Thus, our minisuperspace metric can have three degrees of freedom (ψ_+, ψ_-, α).

With the metric parametrization given in Eq. (4.2.1), we can integrate out the spatial coordinates and then making a change of time coordinate to the "dilaton time" [169] (this equivalent to the Taub gauge [190] in general relativity)

$$d\tau \equiv e^{\bar{\phi}}\ dt \tag{4.2.3}$$

we obtain, from Eq. (3.1.9), the effective string action

$$S = \frac{\lambda_s}{2}\int d\tau \left[-\bar{\phi}'^2 + \beta'^2 + 6\left(\psi_+'^2 + \psi_-'^2\right)\right.$$
$$\left. - e^{-2\bar{\phi}}\left(\hat{V}(\beta, \psi_\pm) + \Lambda + \frac{1}{12}H^2\right)\right], \tag{4.2.4}$$

where, in analogy with [102], we define

$$\beta \equiv \sqrt{3}\alpha, \tag{4.2.5}$$

$$\bar{\phi} \equiv \phi - 3\alpha - \ln\int \frac{d^3x}{\lambda_s^3}. \tag{4.2.6}$$

Here $\bar{\phi}$ is called the shifted dilaton [82]. The potential

$$\hat{V}(\beta, \psi_\pm) \equiv e^{-2\beta/\sqrt{3}}V(\psi_\pm) \tag{4.2.7}$$

is due to the intrinsic spatial curvature, and the form of $V(\psi_\pm)$ depends on the particular Bianchi type. For example, the most complicated and in some sense most general closed Bianchi model of Bianchi type IX, the curvature potential $V(\psi_\pm)$ is given by [74]

$$V(\psi_\pm) = \frac{1}{2}\left[e^{-8\psi_+} + 2e^{4\psi_+}\left(\cosh 4\sqrt{3}\psi_- - 1\right)\right.$$
$$\left. - 4e^{-2\psi_+}\cosh 2\sqrt{3}\psi_-\right]. \tag{4.2.8}$$

We can obtain an equivalent effective action using pseudoscalar axion field h instead of axion H by means of relation (3.3.10). The dual effective action is then

$$S_* = \frac{1}{2\lambda_s^2}\int d^4x\,\sqrt{g}\,e^{-\phi}\left[R + (\nabla\phi)^2 - \Lambda - \frac{1}{2}e^{2\phi}(\nabla\sigma)^2\right]. \tag{4.2.9}$$

This introduces a potential concern about the validity of studying the quantum cosmology of this action as the axion term in the action (4.2.9) is not the same as the H^2 term original action (3.1.9) as it differs by an overall sign. Moreover the ansatz (3.3.10) automatically satisfies the classical equation of motion, but satisfies the closure identity only if the new equation of motion for h derived from the dual effective action is obeyed. To avoid this ambiguity we will integrate out the degree of freedom associated with the classical evolution of the field h (and hence $H_{\mu\nu\lambda}$) to yield an effective potential for the dilaton and metric degrees of freedom due to their interaction with the antisymmetric tensor field.

In order to be consistent with our usage of a homogeneous metric, the energy-momentum tensor for the pseudoscalar axion field h must be homogeneous (see Chapter 3). First choice for the axion field H is solitonic ansatz (3.3.16) which in terms of minisuperspace variables (ψ_+, ψ_-, α) gives

$$H^2 = H_{\mu\nu\lambda}H^{\mu\nu\lambda} = \frac{6q^2}{e^{6\alpha}}, \tag{4.2.10}$$

and $q =$ const. Classical solutions with homogeneous axion field can be generated by applying SL(2,R) ("S-duality") transformations to the homogeneous dilaton-vacuum solutions [55, 58, 157].

However, as mentioned in Chapter 3 it is also possible to have an elementary ansatz for the axion field H in some homogeneous cosmologies [25, 20], so long as its energy-momentum tensor remains homogeneous. This is not consistent with an isotropic FRW cosmology as the gradient of the scalar field picks out a prefered direction, but we will consider the effect of an inhomogeneous axion field in a Bianchi I model [57], and in Kantowski-Sachs cosmologies [16]. In terms of minisuperspace variables (ψ_+, ψ_-, α) we have (compare (3.3.78))

$$H^2 = -\frac{6p^2 e^{2\phi}}{e^{2(\alpha-2\psi_+)}}, \tag{4.2.11}$$

and $p =$ const. Classical solutions with an elementary ansatz can be generated from the homogeneous dilaton-vacuum solutions by O(3,3) (T-duality) transformations [169].

4.3 Isotropic quantum string cosmology

As a simplest case one can consider a spatially flat FRW universe, in which case $V(\psi_\pm) = 0$ in (4.2.8) and $\psi_+ = \psi_- = 0$. The minisuperspace has two degrees of freedom: the scale factor e^α and the dilaton ϕ, or equivalently $\bar\phi$ and β defined in Eqs. (4.2.5) and (4.2.6). Gasperini and Veneziano [102] considered the effective action, neglecting the spatial curvature and the antisymmetric tensor field H, in the form

$$S = \frac{\lambda_s}{2} \int d\tau \left[-\bar\phi'^2 + \beta'^2 - e^{-2\bar\phi}\Lambda \right]. \tag{4.3.1}$$

Including spatial curvature term $-\mathcal{K}/a^2$ ($V(\psi_\pm) = -6\mathcal{K}$) and a homogeneous antisymmetric tensor field given by (4.2.10) is equivalent to introducing a potential term for the field β giving [82]

$$\Lambda \to \lambda(\beta) \equiv \Lambda - 6\mathcal{K}e^{-2\beta/\sqrt{3}} + \frac{1}{2}q^2 e^{-2\sqrt{3}\beta}. \tag{4.3.2}$$

The classical evolution of this system was investigated in [82, 105]. Note that the elementary ansatz is inconsistent with the isotropic FRW model.

The canonical momenta with respect to β and $\bar\phi$ are

$$\Pi_\beta = \lambda_s \beta', \tag{4.3.3}$$

$$\Pi_{\bar\phi} = -\lambda_s \bar\phi'. \tag{4.3.4}$$

Thus the Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2\lambda_s} \left[\Pi_\beta^2 - \Pi_{\bar\phi}^2 + \lambda_s^2 e^{-2\bar\phi}\lambda(\beta) \right], \tag{4.3.5}$$

and the Wheeler-de Witt (WdW) equation is

$$\hat{\mathcal{H}}\Psi = \left[\partial_{\bar\phi}^2 - \partial_\beta^2 + \lambda_s^2 e^{-2\bar\phi}\lambda(\beta) \right] \Psi = 0. \tag{4.3.6}$$

In the absence of any cosmological constant, spatial curvature, or axion field, we have no potential term, $\lambda = 0$, and the solutions are plane waves, $\Psi \sim e^{-ik(\bar\phi \pm \beta)}$, corresponding to classical solutions with constant momenta.

The solutions with a non-vanishing cosmological constant only, $\lambda(\beta) = \Lambda > 0$, are

$$\Psi \sim e^{\mp ik\beta} \bar\psi_k(\bar\phi), \tag{4.3.7}$$

where

$$\bar{\psi}_k(\bar{\phi}) = A_{k1} J_{ik}(\lambda_s \sqrt{\Lambda} e^{-\bar{\phi}}) + A_{k2} Y_{ik}(\lambda_s \sqrt{\Lambda} e^{-\bar{\phi}}), \quad (4.3.8)$$

and J_{ik} and Y_{ik} are Bessel functions of imaginary order. These solutions were discussed in [102, 150].

The presence of a Λ-term, although compatible with T-duality of the spatially flat FRW solutions, breaks the S-duality invariance of the theory [162]. Conversely, the presence of a homogeneous axion field or spatial curvature is compatible with S-duality but breaks T-duality. In the following we consider the case of vanishing Λ-term which leaves S-duality unbroken and allows us to solve explicitly the WdW equation including both the homogeneous axion field and spatial curvature.

4.3.1 Axion-dilaton quantum cosmology in flat space

The case of vanishing dilaton potential ($\Lambda = 0$) but with non-vanishing homogeneous axion and spatial curvature was considered in [161, 163]. Here we have the effective action

$$S = \frac{\lambda_s}{2} \int d\tau \left[-\bar{\phi}'^2 + \beta'^2 - \frac{1}{2} q^2 e^{-2\bar{\phi} - 2\sqrt{3}\beta} \right]. \quad (4.3.9)$$

The potential is a product of $\bar{\phi}$ and β, but we can choose infinitely many different combinations of $\bar{\phi}$ and β as our canonical variables, corresponding to different rotations in the field-space. Here the natural choice is

$$\begin{aligned} \phi &\equiv \bar{\phi} + \sqrt{3}\beta, \\ y &\equiv \sqrt{3}\bar{\phi} + \beta \equiv \sqrt{3}(\phi - 2\ln a - \ln \int \frac{d^3x}{\lambda_s^3}), \end{aligned} \quad (4.3.10)$$

which gives

$$S = \frac{\lambda_s}{4} \int d\tau \left[\phi'^2 - y'^2 - q^2 e^{-2\phi} \right]. \quad (4.3.11)$$

Note that $y \propto -\ln \tilde{a}$ where $\tilde{a}$ is the scale factor in the conformally related Einstein frame.

The action, and thus the solutions, are very similar to that given in [102] for the case of Λ-dilaton cosmology. They are related by the formal transformation (see also notes in [163]):

$$\begin{array}{cccc} & \bar{\phi} \to \phi & \beta \to y & \\ \Lambda \to q^2 & \lambda_s \quad \to & \dfrac{-i\lambda_s}{2} & \tau \to i\tau. \end{array} \tag{4.3.12}$$

It appears that this is just a mathematical relation between different mini-superspace models with limited number of degrees of freedom and it does not reflect any underlying symmetry in the full theory.

The WdW equation (4.3.6) now becomes

$$\hat{\mathcal{H}}\Psi = \frac{1}{\lambda_s}\left[\partial_y^2 - \partial_\phi^2 + \frac{1}{4}\lambda_s^2 q^2 e^{-2\phi}\right]\Psi = 0, \tag{4.3.13}$$

and the separable solution is thus

$$\Psi = e^{-iky}\psi_k(\phi), \tag{4.3.14}$$

where

$$\partial_\phi^2\psi_k + \left(k^2 - \frac{\lambda_s^2 q^2}{4}e^{-2\phi}\right)\psi_k = 0. \tag{4.3.15}$$

This is the wave equation for a particle moving in an effective potential $V(\phi) = \lambda_s^2 q^2 e^{-2\phi}/4$. Note that V is positive, unlike the Λ-dilaton case, as so we have a classically forbidden region, $\phi < \ln|\lambda_s q/2k|$, where $V(\phi) > k^2$.

The general solution is

$$\psi_k(\phi) = C_{k1}I_{ik}(\lambda_s e^{-\phi}q/2) + C_{k2}K_{ik}(\lambda_s e^{-\phi}q/2), \tag{4.3.16}$$

where I_{ik} and K_{ik} are the modified Bessel functions of imaginary order.

As $\phi \to \infty$ we have the familiar classical dilaton cosmology where $k^2 \gg V(\phi)$ and we have plane wave solutions

$$\Psi_\infty \sim e^{-ik(y\pm\phi)} \sim e^{-ik(\sqrt{3}\pm1)(\bar{\phi}\pm\beta)}. \tag{4.3.17}$$

Note that it is the choice of sign of k (rather than the choice of the explicit $+/-$ signs in this equation) that determines the momentum of $\bar{\phi}$:

$$\Pi_{\bar{\phi}}\Psi_\infty = \left(\sqrt{3}\mp1\right)k\Psi_\infty \tag{4.3.18}$$

$$\Pi_\beta\Psi_\infty = \pm\left(\sqrt{3}\pm1\right)k\Psi_\infty. \tag{4.3.19}$$

Hence for k greater or less than zero we are on a pre or post big-bang branch (denoted (+) or (−) branches in [36, 37, 39, 127, 128]) with $\bar{\phi}$ increasing or decreasing respectively.

As $\phi \to -\infty$ we have [1]

$$\Psi_{-\infty} = e^{\pm iky}\sqrt{\frac{e^{\phi}}{\lambda_s q}}\left[\frac{C_{k1}}{\sqrt{\pi}}\exp\left(\frac{\lambda_s q}{2}e^{-\phi}\right) + \sqrt{\pi}C_{k2}\exp\left(-\frac{\lambda_s q}{2}e^{-\phi}\right)\right]. \tag{4.3.20}$$

As we would expect, the canonical momentum with respect to ϕ is purely imaginary in the forbidden region, although the momentum with respect to y is constant $\Pi_y\Psi = i\partial_y\Psi = k\Psi$.

The question now is what boundary conditions should we impose on our solution.

Our proposal is to pick $C_{k1} = 0$, i.e., $\psi_k(\phi) \sim K_{ik}(\lambda_s q e^{-\phi}/2)$ which decays for $\phi \to -\infty$ and is purely in-going into the forbidden region. This is the only possible choice of wavefunction that remains finite as $\phi \to -\infty$ (which is precisely the regime in which the low energy effective action should be applicable).

In the strong coupling limit, $\phi \to \infty$, this leads to a superposition of plane waves

$$\Psi_k^{\infty}(\phi) = K_{ik}(\lambda_s q e^{-\phi}/2)$$
$$= \frac{i\pi}{2\sinh(k\pi)}\left[\left(\frac{q\lambda_s}{4}\right)^{ik}\frac{e^{-ik\phi}}{\Gamma(1+ik)} - \left(\frac{q\lambda_s}{4}\right)^{-ik}\frac{e^{ik\phi}}{\Gamma(1-ik)}\right] \tag{4.3.21}$$
$$= \Psi_k^{\infty(+)} + \Psi_k^{\infty(-)}.$$

This is in agreement with the similar situation for a minimally coupled massless scalar field [135]. Wave packets can then be constructed here by a superposition

$$\Psi(\phi, y) = \int_{-\infty}^{\infty} dk A(k) K_{ik}\left(\lambda_s q e^{-\phi}/2\right)e^{-iky} \tag{4.3.22}$$

with a suitable amplitude $A(k)$.

There is no way to pick boundary conditions for the wavefunction that suppresses the probability either in the strong-coupling limit $\phi \to \infty$ or at the curvature singularity $y \to \infty$, as these regimes are not classically forbidden.

4.3.2 Axion-dilaton quantum cosmology in curved space

If we set $\Lambda = 0$ but include the axion energy density and spatial curvature, we have the effective action

$$S = \frac{\lambda_s}{2} \int d\tau \left[-\bar{\phi}'^2 + \beta'^2 - e^{-2\bar{\phi}} \left(\frac{1}{2} q^2 e^{-2\sqrt{3}\beta} - 6\mathcal{K} e^{-2\beta/\sqrt{3}} \right) \right]. \tag{4.3.23}$$

Using the variables ϕ and y defined by (4.3.10) gives

$$S = \frac{\lambda_s}{4} \int d\tau \left[\phi'^2 - y'^2 - q^2 e^{-2\phi} + 12\mathcal{K} e^{-2y/\sqrt{3}} \right], \tag{4.3.24}$$

and hence the WdW equation becomes

$$\hat{\mathcal{H}}\Psi = \frac{1}{\lambda_s} \left[\partial_y^2 - \partial_\phi^2 + \frac{1}{4} \lambda_s^2 q^2 e^{-2\phi} - 3\lambda_s^2 \mathcal{K} e^{-2y/\sqrt{3}} \right] \Psi = 0. \tag{4.3.25}$$

The separable solution is

$$\Psi = \chi_k(y) \psi_k(\phi), \tag{4.3.26}$$

where $\psi_k(\phi)$ has the same form as given in (4.3.16) and $\chi_k(y)$ is the solution of

$$\partial_y^2 \chi_k + \left(k^2 - 3\lambda_s^2 \mathcal{K} e^{-2y/\sqrt{3}} \right) \chi_k = 0. \tag{4.3.27}$$

The general solution in the case $\mathcal{K} > 0$ is

$$\begin{aligned} \chi_k(y) &= D_{k1} I_{i\sqrt{3}k}(3\lambda_s \sqrt{\mathcal{K}} e^{-y/\sqrt{3}}) \\ &\quad + D_{k2} K_{i\sqrt{3}k}(3\lambda_s \sqrt{\mathcal{K}} e^{-y/\sqrt{3}}). \end{aligned} \tag{4.3.28}$$

If we are to keep the wavefunction finite as $y \to -\infty$, we must again pick $\chi_k \sim K_{i\sqrt{3}k}(\lambda_s 3\sqrt{\mathcal{K}} e^{-y/\sqrt{3}})$.

For $\mathcal{K} < 0$ we have

$$\begin{aligned} \chi_k(y) &= E_{k1} J_{i\sqrt{3}k}(3\lambda_s \sqrt{|\mathcal{K}|} e^{-y/\sqrt{3}}) \\ &\quad + E_{k2} Y_{i\sqrt{3}k}(3\lambda_s \sqrt{|\mathcal{K}|} e^{-y/\sqrt{3}}). \end{aligned} \tag{4.3.29}$$

Note that for $\mathcal{K} > 0$ there is a classically forbidden region,

$$y < \sqrt{3} \ln \left| \frac{\lambda_s \sqrt{3|\mathcal{K}|}}{k} \right|, \tag{4.3.30}$$

where $k^2 < 3\lambda_s^2 \mathcal{K} e^{-2y/\sqrt{3}}$. Note that this excludes the classical late-time "post-big bang" branch where $\phi \to \infty$ and $y \to -\infty$ in a closed model. (In the Einstein conformal frame this is just the recollapse of the universe.) We need $\Lambda \geq 8\mathcal{K}/|q|$ to rescue us from this [82].

For $\mathcal{K} < 0$ there is no classical bound on y, but as in the Λ case investigated in [102] we could scatter off the y-dependent potential.

4.4 Anisotropic quantum string cosmologies

In this section we extend the previous discussion of isotropic Friedmann models to other Bianchi types and Kantowski-Sachs models which include the shear degrees of freedom and anisotropic curvature.

We will show that the form of the wavefunction with respect to the dilaton, $\psi(\phi)$, is unchanged by the spatial geometry as long as S-duality is unbroken and $\Lambda = 0$.

4.4.1 Bianchi I

The simplest anisotropic model with non-vanishing shear is the spatially flat Bianchi type I cosmology where $\hat{V} = 0$. In this anisotropic geometry it is possible to consider either a homogeneous or an inhomogeneous axion field.

Solitonic axion ansatz

Using variables given in (4.3.10) and H^2 given by (4.2.10) the action (4.2.4) is given by

$$S = \frac{\lambda_s}{4} \int d\tau \left[\phi'^2 - y'^2 + 12 \left(\psi_+'^2 + \psi_-'^2 \right) - q^2 e^{-2\phi} - 2\Lambda e^{\phi - y\sqrt{3}} \right]. \tag{4.4.1}$$

The canonical momenta are then

$$\begin{aligned}\Pi_\phi &= \frac{\lambda_s}{2}\phi', \\ \Pi_y &= -\frac{\lambda_s}{2}y', \\ \Pi_+ &= 6\lambda_s\psi'_+, \\ \Pi_- &= 6\lambda_s\psi'_-,\end{aligned} \tag{4.4.2}$$

and the Hamiltonian reads as

$$\mathcal{H} = \frac{1}{\lambda_s}\left[\Pi_\phi^2 - \Pi_y^2 + \frac{1}{12}\left(\Pi_+^2 + \Pi_-^2\right) + \frac{\lambda_s^2}{4}q^2e^{-2\phi} + \frac{\lambda_s^2}{2}\Lambda e^{\phi - y\sqrt{3}}\right]. \tag{4.4.3}$$

Now we can write down WdW equation as

$$\hat{\mathcal{H}}\Psi = \left[\partial_y^2 - \partial_\phi^2 - \frac{1}{12}\left(\partial_+^2 + \partial_-^2\right) + \frac{\lambda_s^2}{4}q^2e^{-2\phi} + \frac{\lambda_s^2}{2}\Lambda e^{\phi - y\sqrt{3}}\right]\Psi = 0. \tag{4.4.4}$$

Because there is no potential term driving the shear, $\psi_\pm$, we can easily give solutions to (4.4.4) with $\Psi \propto e^{-ip_+\psi_+}e^{-ip_-\psi_-}$. However we can only separate the potential for ϕ and y if either $\Lambda = 0$ or $q = 0$. Considering the dilaton-axion theory with $\Lambda = 0$ we obtain

$$\Psi(y, \phi, \psi_+, \psi_-) = e^{-iKy}e^{-ip_+\psi_+}e^{-ip_-\psi_-}\psi_k(\phi), \tag{4.4.5}$$

where K, p_+, p_- are constant momenta. Equation (4.4.4) then reduces to that given in (4.3.15) for $\psi_k(\phi)$ in the isotropic Friedmann model, where

$$k^2 \equiv K^2 - \frac{1}{12}\left(p_+^2 + p_-^2\right), \tag{4.4.6}$$

and the solution for $\psi_k(\phi)$ is given by (4.3.16).

For $K^2 > (p_+^2 + p_-^2)/12$ the solutions $\psi_k(\phi)$, and their interpretation are identical to the isotropic quantum cosmology given in (4.3.16). For $K^2 \leq (p_+^2 + p_-^2)/12$ the order of the modified Bessel functions, ik, is no longer imaginary, and there is no classically allowed regime, as these would correspond to solutions with negative energy density.

Elementary axion ansatz

We also consider the case where the gradient of the axion field is spacelike, and the antisymmetric tensor field strength, H^2, is given by (4.2.11). We introduce rotated fields

$$\begin{aligned} \beta_+ &\equiv \sqrt{\frac{2}{3}}\beta + \sqrt{2}\psi_+ \\ \beta_- &\equiv \sqrt{\frac{1}{3}}\beta - 2\psi_+, \end{aligned} \tag{4.4.7}$$

which allows us to write the action in Eq. (4.2.4) as

$$S = \frac{\lambda_s}{2}\int d\tau \left[-\bar{\phi}'^2 + \beta_+'^2 + \beta_-'^2 + 6\psi_-'^2 - \Lambda e^{-2\bar{\phi}} - \frac{p^2}{2}e^{2\sqrt{2}\beta_+} \right]. \tag{4.4.8}$$

Note that $e^{\beta_+/\sqrt{3}}$ and $e^{\beta_-/\sqrt{3}}$ describe the two scale factors in an axially symmetric Bianchi I model where $\psi_- =$ constant. The inhomogeneous axion leads to an effective potential for β_+ that is orthogonal in field-space to the Λ-potential for the shifted dilaton, $\bar{\phi}$. The classical solutions for an inhomogeneous axion are related by a O(3,3) T-duality transformation to the dilaton-vacuum solutions [57, 169] and thus the equations can be solved analytically.

We can write down the WdW equation as

$$\hat{\mathcal{H}}\Psi = \left[\partial_{\bar{\phi}}^2 - \partial_+^2 - \partial_-^2 - \frac{1}{6}\partial_\psi^2 + \frac{\lambda_s^2}{2}p^2 e^{2\sqrt{2}\beta_+} + \lambda_s^2\Lambda e^{-2\bar{\phi}} \right]\Psi = 0, \tag{4.4.9}$$

with the solution

$$\Psi(\bar{\phi}, \beta_+, \beta_-, \psi_-) = e^{-iK\beta_-}e^{-ip\psi_-}\bar{\psi}_k(\bar{\phi})\chi_m(\beta_+), \tag{4.4.10}$$

where K and p_- are constant momenta,

$$k^2 = m^2 + K^2 + \frac{1}{6}p_-^2\ , \tag{4.4.11}$$

and the solution for $\bar{\psi}_k(\bar{\phi})$ is the same as that for in a Friedmann cosmology given in (4.3.8), while

$$\chi_m(\beta_+) = D_{k1}I_{im/\sqrt{2}}(\lambda_s p e^{\sqrt{2}\beta_+}/2) \ + D_{k2}K_{im/\sqrt{2}}(\lambda_s p e^{\sqrt{2}\beta_+}/2). \tag{4.4.12}$$

Thus the inhomogeneous axion field places an upper bound on the classically allowed value of the scale factor β_+, and in this respect the spacelike axion is very similar to a positive spatial curvature term. If we are to keep the wavefunction finite as $\beta_+ \to \infty$, we must again pick $D_{k1} = 0$.

4.4.2 Bianchi IX

As it was shown in Chapter 3 in a Bianchi type IX cosmology it is only possible to have a solitonic axion ansatz [16, 20, 57], with H^2 given by (4.2.10). Using (4.3.10) the action (4.2.4) becomes

$$S = \frac{\lambda_s}{4}\int d\tau \left[\phi'^2 - y'^2 + 12\left(\psi_+'^2 + \psi_-'^2\right)\right.$$
$$\left. - q^2 e^{-2\phi} - 2e^{-\frac{2}{\sqrt{3}}y} V(\psi_\pm) - 2\Lambda e^{\phi - y\sqrt{3}}\right], \tag{4.4.13}$$

where $V(\psi_\pm)$ is given by (4.2.8). The canonical momenta are then

$$\begin{aligned} \Pi_\phi &= \frac{\lambda_s}{2}\phi', \\ \Pi_y &= -\frac{\lambda_s}{2}y', \\ \Pi_+ &= 6\lambda_s\psi_+', \\ \Pi_- &= 6\lambda_s\psi_-'. \end{aligned} \tag{4.4.14}$$

The Hamiltonian is

$$\mathcal{H} = \frac{1}{\lambda_s}\left[\Pi_\phi^2 - \Pi_y^2 + \frac{1}{12}\left(\Pi_+^2 + \Pi_-^2\right) + \frac{\lambda_s^2}{4} q^2 e^{-2\phi}\right.$$
$$\left. + \frac{\lambda_s^2}{2} e^{-\frac{2}{\sqrt{3}}y} V(\psi_\pm) + \frac{\lambda_s^2}{2}\Lambda e^{\phi - y\sqrt{3}}\right]. \tag{4.4.15}$$

and we can write down the WdW equation:

$$\hat{\mathcal{H}}\Psi = \left[\partial_y^2 - \partial_\phi^2 - \frac{1}{12}\left(\partial_+^2 + \partial_-^2\right) + \frac{\lambda_s^2}{4} q^2 e^{-2\phi}\right.$$
$$\left. + \frac{\lambda_s^2}{2} e^{-\frac{2}{\sqrt{3}}y} V(\psi^\pm) + \frac{\lambda_s^2}{2}\Lambda e^{\phi - y\sqrt{3}}\right]\Psi = 0 \tag{4.4.16}$$

Once again we see that the potential for the dilaton ϕ decouples from the metric degrees of freedom when $\Lambda = 0$ and can write down a separable solution of the form $\Psi \propto \psi_k(\phi)$, where $\psi_k(\phi)$ is given by (4.3.16).

We can only solve analytically with respect to the metric degrees of freedom in the locally rotationally symmetric case where we fix one of the degrees of freedom, $\psi_- = 0$ and we have

$$V(\psi_+) = \frac{1}{2}e^{-8\psi_+} - 2e^{-2\psi_+}. \qquad (4.4.17)$$

We define

$$\begin{aligned} u &\equiv \frac{2}{\sqrt{3}}y + 2\psi_+, \\ v &\equiv \frac{1}{\sqrt{3}}y + 4\psi_+, \end{aligned} \qquad (4.4.18)$$

and the effective action can be written as

$$S = \frac{\lambda_s}{4}\int d\tau \left[\phi'^2 + v'^2 - u'^2 - q^2 e^{-2\phi} - e^{-2v} + 8e^{-u} - 2\Lambda e^{\phi+v-2u}\right]. \quad (4.4.19)$$

The WDW equation becomes

$$\begin{aligned} \hat{\mathcal{H}}\Psi = \Bigg[\partial_u^2 - \partial_\phi^2 - \partial_v^2 + \frac{\lambda_s^2}{4}q^2 e^{-2\phi} + \frac{\lambda_s^2}{4}e^{-2v} \\ - 2\lambda_s^2 e^{-u} + \frac{\lambda_s^2}{2}\Lambda e^{\phi+v-2u}\Bigg]\Psi = 0. \end{aligned} \qquad (4.4.20)$$

In this case, with $\Lambda = 0$, we can solve analytically for

$$\Psi(\phi, u, v) = \varepsilon_l(v)\chi_m(u)\psi_k(\phi), \qquad (4.4.21)$$

with $\psi_k(\phi)$ given by Eq. (4.3.16), and

$$\chi_m(u) = E_{m1}I_{2im}(2\sqrt{2}\lambda_s e^{-u/2}) + E_{m2}K_{-2im}(2\sqrt{2}\lambda_s e^{-u/2}), \qquad (4.4.22)$$

$$\varepsilon_l(v) = F_{l1}I_{il}(\lambda_s e^{-v}/2) + F_{l2}K_{il}(\lambda_s e^{-v}/2), \qquad (4.4.23)$$

with the additional condition for the separation constants

$$k^2 + l^2 = m^2. \qquad (4.4.24)$$

Note that a similar curvature potential $V(\psi_\pm)$ occurs in Bianchi II models where the second term in (4.4.17) is absent. There is then no potential for u and one obtains $\chi_m(u) \propto e^{-imu}$. Classical solutions which approach Bianchi II geometry have recently been considered as a limiting solution in gravitational collapse in string theory [41].

4.4.3 Kantowski-Sachs

The Kantowski-Sachs metric (2.3.1) can be rewritten in terms of minisuperspace variables ψ, β as [186]

$$ds^2 = -dt^2 + e^{2\beta/\sqrt{3}} \left(e^{-4\psi} dr^2 + e^{2\psi} d\Omega_{\mathcal{K}}^2\right), \tag{4.4.25}$$

where $d\Omega_{\mathcal{K}}^2$ is the line element on a 2-sphere with curvature $\mathcal{K} > 0$. For $\mathcal{K} = 0$ this reduces to the LRS Bianchi I model, whereas for $\mathcal{K} < 0$ this is the Bianchi III line element.

Solitonic axion ansatz

With H^2 given by (4.2.10) the effective action in (4.2.4) is

$$S = \frac{\lambda_s}{4} \int d\tau \left[\phi'^2 - y'^2 + 12\psi'^2 - 2\Lambda e^{\phi - y\sqrt{3}} + 4\mathcal{K} e^{-\frac{2}{\sqrt{3}}y - 2\psi} - q^2 e^{-2\phi}\right]. \tag{4.4.26}$$

Introducing u and v analogously to the LRS Bianchi IX model (4.4.18) with $\psi_+ = \psi$, we then have

$$S = \frac{\lambda_s}{4} \int d\tau \left[\phi'^2 + v'^2 - u'^2 - 2\Lambda e^{\phi + v - 2u} + 4\mathcal{K} e^{-u} - q^2 e^{-2\phi}\right]. \tag{4.4.27}$$

The WdW equation becomes

$$\hat{\mathcal{H}}\Psi = \left[\partial_u^2 - \partial_\phi^2 - \partial_v^2 + \frac{\lambda_s^2}{4} q^2 e^{-2\phi} - \lambda_s^2 \mathcal{K} e^{-u} + \frac{\lambda_s^2}{2} \Lambda e^{\phi + v - 2u}\right] \Psi = 0. \tag{4.4.28}$$

This has a separable solution for $\Lambda = 0$, very similar to the Friedmann solution, namely

$$\Psi = e^{-ilv} \chi_m(u) \psi_k(\phi), \tag{4.4.29}$$

with $\psi_k(\phi)$ given by Eq. (4.3.16) and for $\mathcal{K} > 0$

$$\chi_m(u) = D_{m1} I_{2im}(2\lambda_s \sqrt{\mathcal{K}} e^{-u/2}) + D_{m2} K_{2im}(2\lambda_s \sqrt{\mathcal{K}} e^{-u/2}), \qquad (4.4.30)$$

where k, l and m are separation constants subject to the constraint (4.4.24). Note that in addition to the lower bound on ϕ, there is a classically forbidden region, where the momentum for u becomes imaginary when $m^2 < \lambda_s^2 \mathcal{K} e^{-u}$.

For $\mathcal{K} < 0$ we have

$$\chi_m(u) = E_{m1} J_{2im}(2\lambda_s \sqrt{|\mathcal{K}|} e^{-u/2}) + E_{m2} Y_{2im}(2\lambda_s \sqrt{|\mathcal{K}|} e^{-u/2}), \qquad (4.4.31)$$

and there is no classically forbidden region for u.

Elementary axion ansatz

It is also possible to study quantum cosmology with an elementary axion ansatz which leads to an anisotropic tensor field $H_{\mu\nu\lambda}$, in the case of a Kantowski-Sachs model [16], where $H^2 = -6p^2 e^{2\phi+4\psi-2\beta/\sqrt{3}}$. This leads to the effective action

$$S = \frac{\lambda_s}{4} \int d\tau \left[\phi'^2 - y'^2 + 12\psi'^2 - \Lambda e^{\phi - y\sqrt{3}} + 4\mathcal{K} e^{-\frac{2}{\sqrt{3}}y - 2\psi} - p^2 e^{2\phi - 2y/\sqrt{3} + 4\psi} \right]. \qquad (4.4.32)$$

Using the variables (4.4.18) gives

$$S = \frac{\lambda_s}{4} \int d\tau \left[\phi'^2 + v'^2 - u'^2 - \Lambda e^{\phi + v - 2u} + 4\mathcal{K} e^{-u} - p^2 e^{2(\phi + v - u)} \right], \qquad (4.4.33)$$

which does not seem to have separable solutions after quantization except for WKB solutions.

4.5 Boundary conditions

In sections 4.2–4.5 we gave the general solution to the WdW equation for the axion-dilaton action in a spatially curved FRW metric. We have also discussed the solutions for some anisotropic geometries such as Bianchi I, axisymmetric Bianchi IX and Kantowski-Sachs. We differentiated two cases for an appropriate ansatz for the axion field. One case is when the axion field is homogeneous (solitonic ansatz) and this choice is compatible with most of the Bianchi geometries as well as with isotropic Friedmann geometry. Another case is when the gradient of axion field is spacelike (elementary ansatz) and this ansatz is not compatible with some of the Bianchi type geometries. In particular, we used the second ansatz for Bianchi I and Kantowski-Sachs geometries. What makes the latter case very interesting in classical domain is the fact that it prevents isotropization of homogeneous universes at late times.

On the other hand, quantum string minisuperspaces for Kantowski-Sachs geometry should also attract some attention while drawing conclusions about the arrow of time in cosmology. Namely, as shown in [117, 141], the cosmological and thermodynamic arrows point in different directions during the recollapsing phase of the universe and this conclusion was basically drawn on the base of quantum minisuperspaces with Kantowski-Sachs geometry [141]. However, as it is well-known even classical trajectories of Kantowski-Sachs universe are asymmetric (e.g. they start at or near a disklike singularity and evolve into a cigarlike one). This point was emphasized, for instance, in [138] and can also be posed in the context of quantum string cosmology using the solutions of our Section 4.3. In order to study the problem carefully one should perhaps answer the question what happens with scalar perturbations after big-bang (or, more precisely, in post-big-bang era), i.e, whether they grow large if they originally start at their ground state [41, 117]. In principle, the general question to be answered should be: what is the arrow of time in both pre and post-big-bang phase and how it depends on the choice of branch? We have not studied this interesting problem here, leaving it to some future investigations.

The very important problem in cosmology is the boundary conditions. Although our main calculations are just general solutions of some appropriate WDW equations we also tried to impose the boundary conditions in some simpler cases. For instance the tunneling boundary conditions would require us to pick a purely outgoing (post-big-bang) wavefunction for $\bar{\phi} \to -\infty$ which is claimed to be the spirit of [102]. However, the analogy of [102] is not direct since one has a purely ingoing (pre-big-bang) wave function imposed at $\bar{\phi} \to \infty$ there. In our opinion, the superposition of left and right-moving waves at $\bar{\phi} \to -\infty$ is like the "generic" boundary condition example which Vilenkin gives for a collapsing and re-expanding de-Sitter universe in his section 3 of [204], which he says is the same as the Hartle-Hawking boundary conditions.

Since the axion term does not help a pre- to post-big bang transition as it only effects a bounce in ϕ (or β) and not y (or $\bar{\phi}$), then the appropriate boundary conditions to follow scattering approach of [102] would be to pick $\psi(x) \sim K_{ik}(\lambda_s q e^{-x}/2)$ which decays as $x \to -\infty$ (i.e. is purely in-going, into the forbidden region). We imposed this type of boundary conditions in axion-dilaton isotropic string minisuperspace model of our Section 4.3.

In general, one should probably impose boundary conditions at low-energy limit $\bar{\phi} \to -\infty$ rather than in strong-coupling limit $\bar{\phi} \to +\infty$, where the effective action breaks down, and this will be discussed further [76].

One should also notice that spatial curvature with $\mathcal{K} < 0$ could play the same role to a positive Λ. Positive $\mathcal{K}$ actually forbids the classical post-big bang evolution (unless we have a positive Λ to help it to avoid "recollapse").

One seems to have to look for a solution that always vanishes at strong coupling regime ($\bar{\phi} \to \infty$), but this is not possible when negative dilaton potential (negative Λ) is introduced [97]. However, in such a case both pre- and post-big-bang branches are already connected on the classical level – there is no strong coupling regime – there is no classical singularity so quantum mechanically the wave function may also vanish. This confirms the remarks of [106, 113] which say that even quantum mechanically one cannot avoid a singularity in the sense of vanishing of the wave function, unless there is no singularity classically in the region too.

However, the singularity might be understood to be avoided in the sense that the wave function remains finite at zero volume [141] which happens, for instance, when no-boundary boundary conditions are imposed.

So far, the formalism of canonical quantum gravity has been applied to describe a quantum transition form the "pre-big-bang" phase to the "post-big-bang" phase through the singularity [97, 102]. For instance, in the minisuperspace comprising scale factor a and dilaton ϕ (isotropic quantum string cosmology – Section 4.3), a solution to the Wheeler-DeWitt equation was found after imposing boundary conditions in the strong coupling regime $\phi \to \infty$. This solution was interpreted as describing a reflection in minisuperspace through the singularity.

Such an interpretation is, however, tight with respect to the fact of absence of an external time parameter. Being redundant already in the classical theory due to time-reparametrisation invariance, an external time parameter is completely absent from quantum gravity (see, for example, the careful discussion in [9, 214]). A classical time parameter can only emerge as an approximate notion through some Born-Oppenheimer type of expansion scheme [136].

How, then, can the above "transition" be consistently dealt with in quantum cosmology? The choice of boundary conditions as well as the interpretation of the quantum cosmological wave function should refer only to intrinsic variables, i.e. variables which directly occur in the Wheeler-DeWitt equation. In this respect the hyperbolic nature of this equation for such models is particularly important [214]. The purpose here is the presentation of a consistent quantum cosmological scenario along these lines. Instead of referring to an external time, we shall construct wave packets that represent classical trajectories in quantum cosmology. This has been successfully applied before in quantum general relativity [135]. Furthermore, we shall suggest to impose boundary conditions in the region of small scale factor.

In the following we shall first stick to the simple model where only a positive cosmological constant is present [97, 102]. The main conceptual issues can be discussed clearly in this context. We shall then proceed to discuss an example which exhibits turning points in configuration space.

We start with the Wheeler-De Witt (WDW) equation from a common sector effective action of superstring theories (3.1.9) for zero spatial curvature, which contains a dilaton potential similarly to [97, 102]. In fact, quantum cosmology for string models was first studied in [31]. It reads

$$\hat{\mathcal{H}}\Psi \equiv \left[-\partial^2_{\bar{\phi}} + \partial^2_{\beta} - \lambda_s^2 V(\beta,\bar{\phi}) e^{-2\bar{\phi}}\right] \Psi(\beta,\bar{\phi}) = 0, \qquad (4.5.1)$$

where

$$\beta \equiv \sqrt{3}\ln a, \qquad (4.5.2)$$

$$\bar{\phi} \equiv \phi - 3\ln a - \ln\int\frac{d^3x}{\lambda_s^3}, \tag{4.5.3}$$

are redefined variables, $V(\beta, \bar{\phi})$ is the dilaton potential (it is assumed that the volume of three-space is finite).

We first consider the simplest potential which is just given by a positive cosmological constant, $V(\beta, \bar{\phi}) = \Lambda$, and look for a separable solution of the form (compare (4.3.7))

$$\Psi_k(\beta, \bar{\phi}) = e^{-ik\beta}\psi_k(\bar{\phi}) \tag{4.5.4}$$

for all $k \in \mathbf{R}$ [97, 102]. The function $\psi_k(\bar{\phi})$ then obeys the effective equation

$$\left(-\partial_{\bar{\phi}}^2 + V_{eff}(\bar{\phi})\right)\psi_k(\bar{\phi}) = 0, \tag{4.5.5}$$

where the effective potential is given by

$$V_{eff} = -k^2 - \lambda_s^2\Lambda e^{-2\bar{\phi}}. \tag{4.5.6}$$

Since V_{eff} is always negative, there are no classically forbidden regions in the effective sector (as specified by k) of this simple model and therefore no "turning points". (In the full theory, there are no classically forbidden regions, since the kinetic term is indefinite). In Section 4.2 we discussed other possibilities, where classically forbidden regions and turning points exist.

The general solution of (4.5.5) is given in terms of Bessel functions [102],

$$\psi_k(\bar{\phi}) = c_1 J_{+ik}(z) + c_2 J_{-ik}(z), \tag{4.5.7}$$

with $z \equiv \lambda_s\Lambda e^{-\bar{\phi}}$. In the strong coupling limit $\bar{\phi} \to \infty$ $(z \to 0)$ one has

$$\lim_{z\to 0} J_{\pm ik}(z)e^{-ik\beta} \sim e^{-ik(\beta\pm\bar{\phi})}. \tag{4.5.8}$$

In order to get a deeper insight into the problem and to discuss the correct boundary conditions we include a brief discussion of the classical solutions for the string effective action equations [99, 100, 102, 172, 200]. Because of the string duality-symmetry, one obtains various "pre-big-bang" and "post-big-bang" branches, but we shall discuss those which attracted most interest: the expanding accelerated (or superinflationary) "pre-big-bang" branch and the expanding decelerated "post-big-bang" branch, which are classically divided by a singularity [102]. These are given by

(+) $t < 0$ ("pre-big-bang")

$$\beta = \beta_0 - \ln\tanh\left(-\frac{\sqrt{\Lambda}t}{2}\right), \tag{4.5.9}$$

$$\bar{\phi} = \bar{\phi}_0 - \ln\sinh(-\sqrt{\Lambda}t), \tag{4.5.10}$$

(−) $t > 0$ ("post-big-bang")

$$\beta = \beta_0 + \ln\tanh\left(\frac{\sqrt{\Lambda}t}{2}\right), \tag{4.5.11}$$

$$\bar{\phi} = \bar{\phi}_0 - \ln\sinh(\sqrt{\Lambda}t). \tag{4.5.12}$$

These branches are related by the duality symmetry including time-reflection $\beta(t) \to -\beta(-t)$, $\bar{\phi}(t) = \bar{\phi}(-t)$.

Since the canonical momentum with respect to β is given by $\Pi_\beta = -\lambda_s e^{-\bar{\phi}}\dot{\beta} \equiv -k =$ constant, one can express "expansion" by $k = \lambda_s\sqrt{\Lambda}e^{-\bar{\phi}_0} > 0$. The canonical momentum with respect to $\bar{\phi}$ is given by $\Pi_{\bar{\phi}} = \lambda_s e^{-\bar{\phi}}\dot{\bar{\phi}}$. In the strong coupling regime $\bar{\phi} \to \infty$ it reads for the cases (+) and (−), respectively,

$$\Pi_{\bar{\phi}}^{(\pm)} \stackrel{\bar{\phi}\to\infty}{\to} \pm\lambda_s\sqrt{\Lambda}e^{-\bar{\phi}_0} = \pm k. \tag{4.5.13}$$

A distinction between "expanding" and "contracting" has no intrinsic meaning, however, since we can arbitrarily change the sign of $\dot{\beta}$ after re-introducing the lapse-function [171]. In quantum cosmology, where t is fully absent, this becomes even more evident, since no reference phase $\exp(-i\omega t)$ is available, with respect to which solutions could be classified as, e.g., right-moving or left-moving. This is in full analogy to the situation in ordinary quantum cosmology [138, 214, 213].

To make the identity of an expanding solution with a contracting solution explicit, it is more appropriate to discuss the string scenario in the configuration space formed by $(\beta, \bar{\phi})$. Eliminating t in (4.5.9)–(4.5.12), one finds that the trajectories in configuration space are given by

$$\beta = \beta_0 \pm \operatorname{arsinh}(Ke^{\bar{\phi}}) = \beta_0 \pm \ln\left[\left(K + \sqrt{K^2 + e^{-2\bar{\phi}}}\right) e^{\bar{\phi}}\right], \tag{4.5.14}$$

where the plus sign refers to the "pre-big-bang" branch (+) and the minus sign to the "post-big-bang" branch (−), respectively, and a new constant K,

$$K \equiv \frac{k}{\lambda_s \sqrt{\Lambda}} = \pm e^{-\bar{\phi}_0}, \tag{4.5.15}$$

has been introduced. Note that a change of sign of K corresponds to the change of the branch, the above distinction between (+) and (−) thus holding for $K > 0$. This is the case to which we restrict our analysis without loss of generality.

What we called "pre-big-bang" branch ("post-big-bang" branch) is now the upper (lower) branch in configuration space, and the duality transformation transforms $\beta(\bar{\phi}) - \beta_0 \to \beta_0 - \beta(\bar{\phi})$. We note that the qualitative features of the trajectories remain unchanged if we go back to the original configuration space variables β and ϕ, where ϕ is the original dilaton field, see (4.5.3). The two branches are then given by the equation (if we write $\bar{\phi} = \phi - \sqrt{3}(\beta - \beta_0)$)

$$e^{\phi} = |K|^{-1} e^{\pm(\beta-\beta_0)\sqrt{3}} \sinh(\pm\beta \mp \beta_0).$$

Coming back to the solution (4.5.7) of the effective equation (4.5.6) and its limit (4.5.8), one can see that

$$\lim_{\phi \to \infty} \Pi_{\bar{\phi}} J_{\pm ik}(z) = \mp k J_{\pm ik}, \tag{4.5.16}$$

where $\Pi_{\bar{\phi}} = -i\partial_{\bar{\phi}}$. This quantum relation, or more precisely, its analogy with the classical relation (4.5.13) was the key point in [102] to identify the two solutions $J_{\pm ik}$ with the "pre-big-bang" (J_{-ik}) and the "post-big-bang" (J_{+ik}) solutions, respectively.

As we have argued before, however, such a distinction has no intrinsic meaning. One can only talk about plane waves travelling with respect to the "intrinsic time" β, distinguishing small β and large β, but not the "pre-big-bang" and the "post-big-bang".

In order to gain further insight, we try to construct wave packets following the classical trajectories in configuration space given by (4.5.14). For the sake of this purpose it is convenient to study first a WKB approximation to (4.5.6). Since there are no classically forbidden regions, one has for all values of $\bar{\phi}$,

$$\psi_k(\bar{\phi}) \sim (-V_{eff})^{-\frac{1}{4}} \left[\exp\left(i \int \sqrt{-V_{eff}} d\bar{\phi} \right) + C \exp\left(-i \int \sqrt{-V_{eff}} d\bar{\phi} \right) \right], \tag{4.5.17}$$

where C is a constant, and "exp(+)" refers to "pre-big-bang", while "exp(−)" to "post-big-bang". The total WKB phase ($\psi_k \sim e^{iS_k}$) is then

$$S_k^{(\pm)}(\beta, \bar{\phi}) = -k\beta \pm s_k(\bar{\phi}), \tag{4.5.18}$$

where

$$s_k \equiv \int^{\bar{\phi}} \sqrt{k^2 + \lambda_s^2 \Lambda e^{-2\tilde{\phi}}} d\tilde{\phi}. \tag{4.5.19}$$

This integral can be solved exactly to give

$$s_k = \lambda_s \sqrt{\Lambda} \left\{ K \mathrm{arsinh}(K e^{\bar{\phi}}) - \sqrt{K^2 + e^{-2\bar{\phi}}} \right\}. \tag{4.5.20}$$

By the principle of constructive interference [135], the classical solutions are found through

$$\frac{\partial S_k^{(\pm)}}{\partial k} = -\beta \pm \frac{\partial s_k}{\partial k} = 0, \tag{4.5.21}$$

leading to (4.5.14) for the classical trajectories in configuration space. After rescaling $S_k^{(\pm)} \to S_k^{(\pm)} / \lambda_s \sqrt{\Lambda}$ we have for the total WKB phase (4.5.18)

$$S_k^{(\pm)} = -K\beta \pm K \mathrm{arsinh}(K e^{\bar{\phi}}) \mp \sqrt{K^2 + e^{-2\bar{\phi}}}. \tag{4.5.22}$$

In order to calculate wave packets for the two solutions (4.5.22) we take a Gaussian concentrated around $\bar{K} > 0$:

$$A_k = \pi^{-\frac{1}{4}} b^{-\frac{1}{2}} \exp\left[-\frac{1}{2b^2} (K - \bar{K})^2 \right], \tag{4.5.23}$$

and consider the superposition

$$\Psi^{(\pm)}(\beta, \bar{\phi}) = \int_{-\infty}^{\infty} dK A_k \frac{e^{iS_k^{(\pm)}}}{(-V_{eff})^{\frac{1}{4}}}. \tag{4.5.24}$$

If the width b of the Gaussian is small enough, A_k is concentrated around $K \approx \bar{K}$, and therefore the integral (4.5.24) can be evaluated by expanding $S_k^{(\pm)}$ up to quadratic order in $K - \bar{K}$. Then,

$$iS_k^{(\pm)} = -iK\beta \pm iK\text{arsinh}(\bar{K}e^{\bar{\phi}}) \mp i\sqrt{\bar{K}^2 + e^{-2\bar{\phi}}} \pm i\frac{(K-\bar{K})^2}{2\sqrt{\bar{K}^2 + e^{-2\bar{\phi}}}} + \dots . \tag{4.5.25}$$

Inserting this into (4.5.24) and evaluating the resulting Gaussian integral, we have (choosing $\beta_0 = 0$ for simplicity)

$$\Psi^{(\pm)}(\beta, \bar{\phi}) = \sqrt{\frac{2}{\pi b}}\frac{1}{B}\left(\bar{K}^2 + e^{-2\bar{\phi}}\right)^{-\frac{1}{4}} \exp\left[-i\bar{K}\left(\beta \mp \text{arsinh}(\bar{K}e^{\bar{\phi}})\right)\right.$$
$$\left.\mp\ i\sqrt{\bar{K}^2 + e^{-2\bar{\phi}}}\ \right] \times \exp\left[\frac{1}{2B^2}\left(-\beta \pm \text{arsinh}(\bar{K}e^{\bar{\phi}})\right)^2\right], \tag{4.5.26}$$

where

$$B^2 = \frac{1}{b^2} \mp \frac{i}{\sqrt{\bar{K}^2 + e^{-2\bar{\phi}}}}. \tag{4.5.27}$$

It is obvious that $|\Psi^{(\pm)}|^2$ is peaked around the classical trajectories (4.5.14) in configuration space. The absolute square of the width B is given by

$$|B|^2 = \frac{1}{b^2}\sqrt{1 + \frac{b^4}{\bar{K}^2 + e^{-2\bar{\phi}}}}, \tag{4.5.28}$$

so we have a very "mild spreading" of the packet.

We consider now packets from exact solutions. The strong coupling limit $\bar{\phi} \to \infty$ $(z \to 0)$ was already performed in (4.5.8), while in the low energy limit $\bar{\phi} \to -\infty$ $(z \to \infty)$ we have

$$J_{\pm ik}(z) = \sqrt{\frac{2}{\pi z}}\cos\left(z \mp \frac{\pi i k}{2} - \frac{\pi}{4}\right) + \dots \propto \frac{1}{2}e^{\pm\left(\frac{\pi k}{2} - \frac{i\pi}{4}\right)}\left(e^{iz} + ie^{-iz\mp\pi k}\right). \tag{4.5.29}$$

The corresponding wave packets read

$$\Psi^{(\pm)}(\beta, \bar{\phi}) = \int_{-\infty}^{\infty} dK A_k e^{-ik\beta} J_{\mp ik}(z). \tag{4.5.30}$$

Following the discussion of [102] we note that after taking, for instance, the "pre-big-bang" solution $J_{-ik}(z)$ for $\bar{\phi} \to -\infty$ ($z \to \infty$), one has a superposition of (+) and (−) solutions (cf. (4.5.29)),

$$J_{-ik}(z) \propto e^{iz} + ie^{-iz}e^{\pi k} \equiv (-) + (+), \tag{4.5.31}$$

and therefore the relative probability between (+) and (−) is

$$\frac{\mid \Psi^{(-)}_{\bar{\phi}\to-\infty} \mid^2}{\mid \Psi^{(+)}_{\bar{\phi}\to-\infty} \mid^2} = e^{-2\pi k}. \tag{4.5.32}$$

However, in order to have a sensible wave packet, k should be concentrated around $k \gg 1$. This means that a "transition" into the (−) component for $\bar{\phi} \to -\infty$ could only be interpreted as an extremely unlikely quantum effect in that region, but not as a transition into the other semiclassical component as represented by a wave packet. Roughly speaking, the (−) component does not correspond to a "classical" trajectory if J_{-ik} is chosen as the exact solution.

To achieve interference between (+) and (−) wave packets, one must really superpose both packets,

$$\Psi = \alpha_1 \Psi^{(+)} + \alpha_2 \Psi^{(-)}, \tag{4.5.33}$$

i.e., choose

$$\Psi^{(\pm)}(\beta, \bar{\phi}) = \int_{-\infty}^{\infty} dK A_k e^{-ik\beta} \left[\alpha_1 J_{-ik}(z) + \alpha_2 J_{+ik}(z)\right] \tag{4.5.34}$$

with complex coefficients α_1 and α_2. This happens, for example, if boundary conditions are imposed in the weak coupling limit $\bar{\phi} \to -\infty$ instead of the strong coupling limit $\bar{\phi} \to \infty$, in contrast to [102]. It is important for us that this is also the region where the effective theory can be trusted. A superposition of J_{+ik} and J_{-ik} which corresponds to the (+) solution for $\bar{\phi} \to -\infty$ (compare (4.5.31)) is the Hankel function

$$H^{(2)}_{ik}(z) \overset{z\to\infty}{\sim} \sqrt{\frac{2}{\pi z}} e^{-iz - \frac{\pi k}{2} + \frac{i\pi}{2}}. \tag{4.5.35}$$

Since

$$H_{ik}^{(2)}(z) = J_{ik}(z) - iN_{ik}(z) = (1 - \coth k\pi)\, J_{ik} + \frac{J_{-ik}}{\sinh k\pi}, \qquad (4.5.36)$$

one finds that $H_{ik}^{(2)}(z)$ approaches in the strong coupling limit $\bar{\phi} \to \infty$ ($z \to 0$) the following asymptotic behaviour:

$$H_{ik}^{(2)}(z) \to \frac{1}{\sinh k\pi}\left[\frac{e^{-k\pi}}{\Gamma(1+ik)}\left(\frac{z}{2}\right)^{ik} + \frac{1}{\Gamma(1-ik)}\left(\frac{z}{2}\right)^{-ik}\right]. \qquad (4.5.37)$$

The corresponding “transition factor” from (+) to (−) would then again be given by $e^{-2\pi k}$, but this time a second semiclassical component is indeed present. This is a generic feature if boundary conditions are imposed at $\bar{\phi} \to -\infty$: since the classical solutions overlap in this region, one finds in general a superposition of wave packets for $\bar{\phi} \to \infty$.

Let us make some general remarks about the boundary conditions in quantum string cosmology. If more than two degrees of freedom are present (like for homogeneous geometries of our Section 4.4), the Wheeler-DeWitt equation is, at least for perturbations of Friedmann-type spaces [103], hyperbolic with respect to β. One would thus expect to impose boundary conditions (Cauchy data) at $\beta =$ constant (or $a =$ constant).

As long as one considers only minisuperspace degrees of freedom, the wave packets are just timeless wave tubes. A semiclassical time parameter, as well as the concept of a direction of time can only be defined if a huge number of further degrees of freedom (“higher multipoles”) is present. A semiclassical time parameter emerges if the “background wave function” is in a WKB state [136]. Technically this is achieved by a Born-Oppenheimer type of expansion scheme, with the expansion parameter given by λ_s in the present case. It yields $\partial/\partial t \equiv \nabla S \cdot \nabla$ for background states $\psi_0 \approx e^{iS}$. A time direction then emerges from thermodynamic considerations if one starts from an uncorrelated state for $\beta \to -\infty$. Such an “initial condition” is facilitated by the fact that the potential term in the Wheeler-DeWitt equation vanishes in this limit (except for the dilaton part). Such an initial state can, for example, be of the form $\Psi = \psi_0(\beta, \phi)$, independent of other degrees of freedom (see [54] for the case of quantum general relativity). With increasing values of β, a correlated state would emerge, since the potential now depends explicitly on the higher multipoles. This in turn, leads to decoherence and increasing entropy for the background part (β, ϕ) [104, 138].

For higher values of β one will then enter the semiclassical regime. One will then get, for example, a state of the form

$$\Psi \approx \alpha_1 e^{iS^{(+)}(\beta,\phi)} \chi^{(+)}(\beta, \phi, \{x_\lambda\}) + \alpha_2 e^{iS^{(-)}(\beta,\phi)} \chi^{(-)}(\beta, \phi, \{x_\lambda\}),$$

where $\{x_\lambda\}$ symbolically denotes all higher multipoles. It is then a quantitative question whether there will be also decoherence between these two components, in addition to the decoherence for each single component. It has been argued that there are regions in the (β, ϕ)-plane (concentrated towards negative dilaton values) where decoherence is ineffective [160]. It would nevertheless then be inappropriate to imagine this as a "transition" from one semiclassical component into the other, since the semiclassical approximation breaks down in such a region, so that no time parameter exists there. There exists thus no classical causal relationship between these branches. This makes it very hard to solve the "graceful exit problem" in this framework, and one has to envisage alternatives such as in [81, 96].

Now, let us discuss briefly some other possibilities for the dilaton potential $V(\beta, \bar{\phi})$ in the WDW equation (4.5.1) in order to gain insight into the problem of boundary conditions in other situations. First, we adopt (though hardly justified by string theory) the negative dilaton potential [97]

$$V(\beta, \bar{\phi}) = -V_0 e^{4\bar{\phi}} \qquad (V_0 > 0) \tag{4.5.38}$$

in (4.5.1), which allows one to find the separable solution (4.5.4) with $\psi_k(\bar{\phi})$ obeying the effective equation (4.5.5). But now the effective potential is different from that of (4.5.6) and reads

$$V_{eff} = -k^2 + \lambda_s^2 V_0 e^{2\bar{\phi}}. \tag{4.5.39}$$

The potential (4.5.39) leads to the existence of classically forbidden regions and "turning points". The key point of such a model is that the "pre-big-bang" and the "post-big-bang" branches are already connected at the classical level. It is, however, interesting that, in contrast to most situations in ordinary cosmology, it is not the scale factor a, but the shifted dilaton $\bar{\phi}$ which has a turning point. Another point is that the strong coupling limit $\bar{\phi} \to \infty$ is classically forbidden.

The corresponding classical self-dual solution [97] is given by

$$\bar{\phi} = -\frac{1}{2} \ln \left(\frac{V_0^{\frac{1}{2}}}{L^2} + L^2 V_0^{\frac{1}{2}} t^2 \right), \tag{4.5.40}$$

$$\beta = \beta_0 + L^2 t + \sqrt{1 + L^4 t^2}, \tag{4.5.41}$$

where

$$L = \frac{k}{\lambda_s V_0^{1/4}}. \tag{4.5.42}$$

This solution is nonsingular at $t = 0$, and the evolution of the scale factor seems to describe a transition between the "pre-big-bang" accelerated branch and the "post-big-bang" decelerated branch of singular solutions like (4.5.9)–(4.5.12). After eliminating t from (4.5.40)–(4.5.41) we find for the evolution equation in configuration space $(\beta, \bar{\phi})$ the equation

$$\beta - \beta_0 = \pm \mathrm{arcosh} \frac{e^{\bar{\phi}_0}}{e^{\bar{\phi}}}, \tag{4.5.43}$$

where we have defined

$$e^{\bar{\phi}_0} \equiv \frac{L}{V_0^{1/4}}. \tag{4.5.44}$$

The trajectory (4.5.43) cannot be divided into two branches which could be naturally interpreted as describing "pre-" or "post-big-bang" branches. In particular, the shifted dilaton has a "turning point" for $\bar{\phi} = \bar{\phi}_0$. This equation looks similar to the corresponding equation in the case of a massless scalar field in ordinary cosmology [135], except that the roles of field and scale factor are interchanged. For this reason we have here a "turning point" for the dilaton. As discussed above, β plays the role of an intrinsic time variable.

A sensible boundary condition would then be to have a wave packet in the small β-region concentrated around a large negative value for the dilaton. This would then lead to a wave packet concentrated around the trajectory (4.5.43). We emphasise again that this can only be interpreted as representing *one* cosmological solution. This solution can be labelled "expanding" only after a condition of low entropy is imposed for $\beta \to -\infty$ in the sense discussed above.

It is also intersesting to notice that there is also a self-dual solution which connects smoothly the "pre-big-bang" decelerated branch with the "post-big-bang" accelerated branch with the same (4.5.39) and the opposite sign before $L^2 t$ in (4.5.40), which leads to the same evolution equation in configuration space (4.5.43).

An interesting model with a turning point in β is obtained by taking into account a positive curvature term in the effective action [82]. We shall not, however, include a discussion of this model here, since the essential conceptual features remain unchanged.

Chapter 5

M-theory cosmology

5.1 M-theory and Hořava-Witten theory

From Chapter 3 we know that pre-big-bang inflation is the result of the admission of the cosmological solutions for the common bosonic sector of superstring effective actions 3.1.9. It was also said that in view of duality symmetry that this bosonic action together with all other superstring effective actions are not necessarily the right description of physics at strong coupling, where string coupling parameter g_s (1.1.3) is large. It seems that at strong coupling regime the physics is 11-dimensional and can be described by a new theory which is called provisionally M-theory (see Fig. 1.1). It is important that 11-dimensional supersymmetric theory which includes gravity is known since the 70s and it is the theory called supergravity. Supergravity is believed to be the low-energy limit of M-theory [212]. The bosonic sector of supergravity contains a three-form potential A_3 and the graviton. The action reads as [66, 67, 68]

$$S_{\rm M} = \frac{1}{16\pi G_{11}} \left(\int d^{11}x \sqrt{-g_{11}} \left[R_{11} - \frac{1}{48} F_4^2 \right] + \frac{1}{6} \int A_3 \wedge F_4 \wedge F_4 \right), \quad (5.1.1)$$

where G_{11} is 11-dimensional Newton constant, R_{11} is 11-dimensional Ricci scalar, $F_4 = dA_3$, and the Chern–Simons term arises as a direct consequence of the ($N = 1$) supersymmetry (compare (3.1.3)).

It was shown that the compactification of $N = 1$, $D = 11$ supergravity on a circle, S^1, resulted in the type IIA supergravity theory [48, 122] which was interpreted as the strongly coupled limit of the type IIA superstring (3.1.3) (with $N = 2$ supersymmetries) in terms of an 11-dimensional theory [212]. This correspondence gave Hořava and Witten [120, 121] the idea that one can also compactify eleven-dimensional supergravity on a S^1/Z_2 orbifold (which is a unit interval I) in order to get a heterotic theory with only $N = 1$ supersymmetry. In other words, they proved that the 10-dimensional $E_8 \times E_8$ theory results from an 11-dimensional theory compactified on the orbifold $R^{10} \times S^1/Z_2$ in the same way as the type IIA theory results from an 11-dimensional theory compactified on $R^{10} \times S^1$. This identified strongly coupled limit of heterotic $E_8 \times E_8$ theory as 11-dimensional supergravity compactified on an orbifold. Apart from that, the idea is that gravity propagates in *all eleven dimensions* (see e.g. [177]) while E_8 gauge fields are *restricted* only to 10-dimensional orbifold fixed planes as shown in Fig. 5.1. Randall and Sundrum [179, 180] developed similar to Hořava-Witten scenario which was mainly motivated by the hierarchy problem in particle physics [5, 6, 52, 125, 167]. As a result, they obtained a 5-dimensional spacetime (bulk) with Z_2 symmetry with two/one 3-brane(s) embedded in it, to which all the gauge interactions are confined. In one-brane scenario [179], the brane appears at the $y = 0$ position, where y is an extra (orbifold) dimension, and the 5-dimensional spacetime is an anti-deSitter space with negative 5-dimensional cosmological constant. The extra dimension can be infinite due to the exponential "warp" factor in the metric. In general, not only extra spatial dimensions but also extra time dimensions are allowed [80]. Cosmological solutions in both Hořava-Witten scenario and Randall-Sundrum scenario can both be called *brane universes*, although the origin of Hořava-Witten theory as a candidate for M-theory justifies the name M-theory cosmology for Hořava-Witten cosmological solutions. In this book we will not be discussing very much of Randall-Sundrum models, restricting ourselves only to Hořava-Witten models.

As pictured in Fig. 5.1, $y = x^{11}$ is an orbifold coordinate with $y \in [-\pi\lambda, \pi\lambda]$, $\lambda =$ const. and the orbifold fixed planes are at $y = 0, \pi\lambda$. The Z_2 symmetry acts as $y \to -y$. The eleven dimensional action for such a theory contains a supergravity part S_{SUGRA} and Yang-Mills part S_{YM} which is

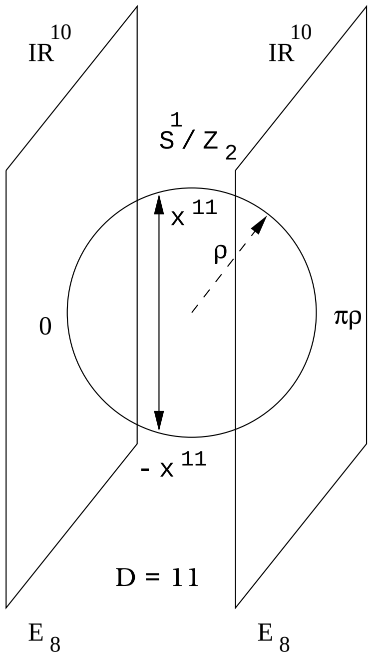

Figure 5.1: Hořava-Witten model

composed of the two E_8 Yang-Mills theories on ten-dimensional orbifold planes as follows

$$S_{SUGRA} = -\frac{1}{2\kappa_1^2 1}\int_{M^{11}}\sqrt{-g_{11}}\Big[R_{11}+\frac{1}{24}G_{IJKL}G^{IJKL} + \frac{\sqrt{2}}{1728}\epsilon^{I_1\ldots I_{11}}C_{I_1I_2I_3}G_{I_4\ldots I_7}G_{I_8\ldots I_{11}}\Big], \quad (5.1.2)$$

and

$$S_{YM} = -\frac{1}{8\pi\kappa_{11}^2}\left(\frac{\kappa_{11}}{4\pi}\right)^{\frac{2}{3}}\int_{M_{10}^{(1)}}\sqrt{-g_{10}}\left[tr\left(F^{(1)}\right)^2-\frac{1}{2}trR^2\right] - \frac{1}{8\pi\kappa_{11}^2}\left(\frac{\kappa_{11}}{4\pi}\right)^{\frac{2}{3}}\int_{M_{10}^{(2)}}\sqrt{-g_{10}}\left[tr\left(F^{(2)}\right)^2-\frac{1}{2}trR^2\right]. \quad (5.1.3)$$

In (5.1.2) and (5.1.3) $I, J, K, \ldots = 0, \ldots, 11$ while $\bar{I}, \bar{J}, \bar{K}, \ldots = 0, \ldots, 9$ and $M_{10}^{(i)}$ (i = 1,2) are ten-dimensional manifolds orthogonal to the orbifold. The $F_{\bar{I},\bar{J}}^{(i)}$ are the two gauge field strengths and C_{IJK} is the three-form potential giving the field strength $G_{IJKL} = 24\partial_{[I}C_{JKL]}$.

Following [159] one is then able to reduce 11-dimensional theory (5.1.2)–(5.1.3) into the 5-dimensional effective theory by compactifying on the deformed Calabi-Yau manifold (three-fold) X. That means we start with manifold $M^{11} = M^4 \times X \times S^1/Z_2$, where M^4 is 4-dimensional spacetime. As shown in Fig. 5.2 there are two copies of M^4 – on each of them there is a gauge group $H^{(i)}$ and $N = 1$ supersymmetry. The 5-dimensional theory contains gravity multiplet $(g_{\mu\nu}, \mathcal{A}_\mu,\ \psi_\mu^i)$, where $\mathcal{A}_\mu$ is a five-dimensional vector field and ψ_μ^i are gravitini. Other fields are $q \equiv (V, \sigma, \xi, \bar{\xi}, \zeta^i)$, where V is modulus field (dilaton) associated with the volume of the Calabi-Yau space, ξ a complex Ramond-Ramond scalar , σ is a dual field to the 3-form $C_{\mu\nu\rho}$ and ζ^i are the fermions. The indices $i, j = 0, 1, 2, 3, \mu, \nu = 0, 1, 2, 3, 5$ here.

The full 5-dimensional effective action reads as

$$S_5 = S_{grav} + S_{hyper} + S_{boundary}, \tag{5.1.4}$$

where

$$S_{gravit} = -\frac{2}{\kappa_5^2}\int_{M_5}\sqrt{-g_5}\left[R^5 + \frac{3}{2}\left(\mathcal{F}_{\mu\nu}\right)^2 + \frac{1}{\sqrt{2}}\epsilon^{\mu\nu\rho\sigma\epsilon}\mathcal{A}_\mu\mathcal{F}_{\nu\rho}\mathcal{F}_{\sigma\epsilon}\right], \tag{5.1.5}$$

$$S_{hyper} = -\frac{1}{2\kappa_5^2}\int_{M_5}\sqrt{-g_5}\left[4h_{\mu\nu}\nabla_{\alpha_0}q^\mu\nabla^\alpha q^\nu + \frac{\alpha^2}{3}V^{-2}\right], \tag{5.1.6}$$

$$S_{boundary} = \frac{\sqrt{2}}{\kappa_5^2}\left[\mp\sum_{i=1}^{2}\int_{M_4^{(i)}}\sqrt{-\tilde{g}_4}\alpha_0 V^{-1}\right]$$

$$- \frac{1}{8\pi\kappa_5^2}\left(\frac{\kappa_{11}}{4\pi}\right)^{\frac{2}{3}}\sum_{i=1}^{2}\int_{M_4^{(i)}}\sqrt{-\tilde{g}_4}V tr\left(\mathcal{F}_{\mu\nu}^{(i)}\right)^2, \tag{5.1.7}$$

and $\kappa_{11}^2 = v\kappa_5^2$ and $v = \text{const}$.

It is important that the size of the orbifold is much bigger than the radius of the Calabi-Yau space and we further reduce a 5-dimensional effective theory

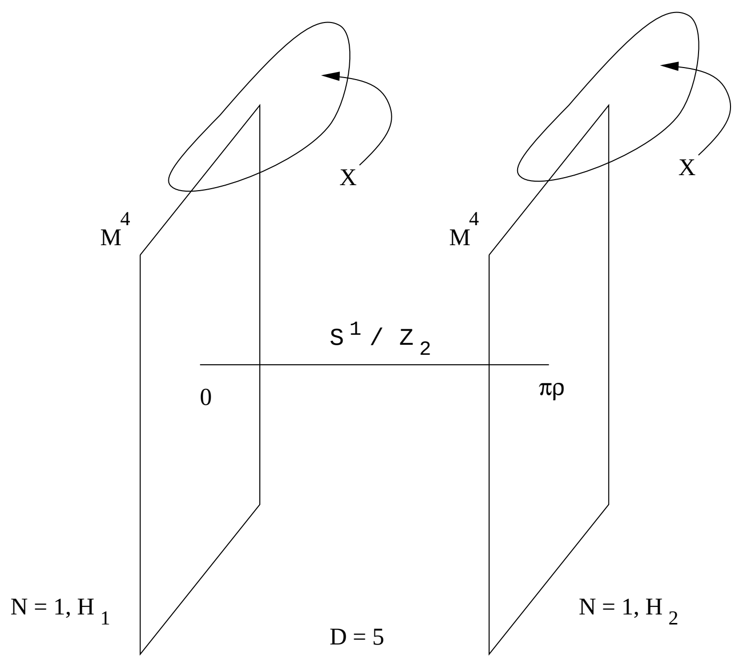

Figure 5.2: Hořava-Witten model compactified on a Calabi-Yau manifold

to [159, 181]

$$S = \int_{M_5} \sqrt{-g_5}\left(\frac{1}{2}R^5 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{6}\alpha_0^2 e^{-2\sqrt{2}\phi}\right)$$
$$\mp \sqrt{2}\sum_{i=1}^{2}\int_{M_4^{(i)}} \sqrt{-\tilde{g}_4}\alpha_0 e^{-\sqrt{2}\phi}, \qquad (5.1.8)$$

where $M_4^{(1)}, M_4^{(1)}$ are orbifold fixed planes, $\phi = 1/\sqrt{2}\ln V$ is a scalar field (dilaton) which parametrizes the radius of Calabi-Yau space and $\tilde{g}_{ij}, i, j = 0, 1, 2, 3$ is the pull-back of 5-dimensional metric onto $M_4^{(1)}$ and $M_4^{(1)}$. In the action (5.1.5) we dropped other important fields like p-form fields, gravitini, RR scalar and fermions.

The field equations which come from the action (5.1.8) are given by [159, 181]

$$
\begin{aligned}
R_\mu^\nu &= \nabla_\mu \nabla^\nu \phi + \frac{\alpha_0^2}{9} g_\mu^\nu e^{-2\sqrt{2}\phi} \\
&+ \sqrt{2}\alpha_0 e^{-\sqrt{2}\phi} \sqrt{\frac{\tilde{g}}{g}} \tilde{g}^{ij} \left[g_{i\mu} g_j^\nu - \frac{1}{3} g_{i\sigma} g_j^\sigma \right] [\delta(y) - \delta(y - \pi\lambda)], \quad (5.1.9)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \partial^\mu \phi \right) &= -\frac{\sqrt{2}}{3} \alpha_0^2 e^{-2\sqrt{2}\phi} \\
&+ 2\alpha \sqrt{\frac{\tilde{g}}{g}} e^{-\sqrt{2}\phi} [\delta(y) - \delta(y - \pi\lambda)], \quad (5.1.10)
\end{aligned}
$$

where $\phi = 1/\sqrt{2} \ln V$ and V is a scalar field measuring the deformation of the Calabi-Yau space, $g_{\mu\nu}$ is the five dimensional metric tensor while g_{ij} is the four dimensional metric which denotes the pull-back of the metric on five-dimensional manifold M_5 onto the orbifold fixed four-dimensional manifolds $M_4^{(1)}$ and $M_4^{(2)}$. In (5.1.9)–(5.1.10) we have neglected the terms which come from the three-form on the Calabi-Yau space. Actually, they will not make any qualitative change in our discussion – this is on the same footing as it was the case in pre-big-bang models [17, 56]. In (5.1.9)–(5.1.10) the terms involving delta functions arise from the stress energy on the boundary planes [206].

5.2 Hořava-Witten cosmologies

In this Section we will consider the models in which M-theory is compactified on an orbifold and then reduced to four dimensions using Calabi-Yau manifold [159, 181]. If we assume that the size of the orbifold is bigger than the size of the Calabi-Yau space, then there was a period in the history of the universe during which the universe was 5-dimensional. That means we can consider the cosmological models for which the fifth coordinate is an orbifold, while the remaining four coordinates are such that the three-space is homogeneous.

We consider only isotropic Friedmann and homogeneous Bianchi type I or IX geometries. The main objective is to study Kasner-type solutions of Bianchi type I field equations and Kasner asymptotic states (as a result of Kasner-to-Kasner transitions) of Bianchi type IX field equations. The form of these solutions allows us to find out whether there is a possibility for chaotic behaviour in these Hořava-Witten cosmologies. Similar question was addressed in our Section 3.4 (see [17]) for pre-big-bang cosmology with the answer that only finite number of chaotic oscillations are possible.

As we know from Chapter 3, the vacuum BIX homogeneous cosmology in general relativity is chaotic [30]. An infinite number of oscillations of the orthogonal scale factors occurs in general on any finite interval of proper time including the singularity at $t = 0$.

If a minimally coupled, massless scalar field (e.g. the inflaton) is admitted, the situation changes. Only a finite number of spacetime oscillations can occur before the evolution is changed into a state in which all directions shrink monotonically to zero as the curvature singularity is reached and the oscillatory behaviour ceases [29]. This is also the case in 4-dimensional pre-big-bang cosmology where the role of a scalar field is played by the dilaton [17]. On the other hand, 5-dimensional vacuum Einstein solutions of Bianchi type IX do not allow chaos to occur either [24, 114, 123]. The point is that the fifth dimension plays effectively the role of a scalar field in scalar field cosmologies and stops chaotic oscillations. In 5-dimensional Hořava-Witten cosmology the situation is in some ways analogous to both of the above cases, and therefore some new interesting observations can be made.

Following [159] we consider cosmological models of the form

$$ds_5^2 = -N^2(\tau, y)d\tau^2 + ds_3^2 + d^2(\tau, y)dy^2, \tag{5.2.1}$$

where

$$ds_3^2 = a^2(\tau, y)(\sigma^1)^2 + b^2(\tau, y)(\sigma^2)^2 + c^2(\tau, y)(\sigma^3)^2 \tag{5.2.2}$$

is a homogeneous Bianchi type IX 3-metric, and the orthonormal forms $\sigma^1, \sigma^2, \sigma^3$ are given by (3.4.2)–(3.4.4). If $\sigma^i = dx^i$ $(i = 1, 2, 3)$ we deal with homogeneous

Bianchi I model and if we additionally take $a = b = c$, then we deal with a flat Friedmann cosmology. Similarly as in [159, 181] we will look for separable solutions of the form

$$\begin{aligned} N(\tau, y) &= n(\tau)\tilde{a}(y), \\ a(\tau, y) &= \alpha(\tau)\tilde{a}(y), \\ b(\tau, y) &= \beta(\tau)\tilde{a}(y), \\ c(\tau, y) &= \gamma(\tau)\tilde{a}(y), \\ d(\tau, y) &= \delta(\tau)\tilde{d}(y), \\ V(\tau, y) &= \varepsilon(\tau)\tilde{V}(y). \end{aligned} \tag{5.2.3}$$

The nonzero components of the field equations read as (in this Chapter an overdot means a derivative with respect to time τ and a prime means a derivative with respect to an orbifold coordinate y)

$$\frac{\tilde{a}^2}{\tilde{d}^2}\left[-\frac{\tilde{a}''}{\tilde{a}} - \frac{\tilde{a}'}{\tilde{a}}\left(3\frac{\tilde{a}'}{\tilde{a}} - \frac{\tilde{d}'}{\tilde{d}}\right) - \frac{1}{9}\alpha_0^2\frac{\delta^2}{\varepsilon^2}\frac{\tilde{d}^2}{\tilde{V}^2} - \frac{\sqrt{2}}{3}\alpha_0\frac{\delta}{\varepsilon}\frac{\tilde{d}}{\tilde{V}}\left(\delta(y) - \delta(y - \pi\lambda)\right)\right]$$
$$= \frac{\delta^2}{n^2}\left[\frac{\dot{n}}{n}\left(\frac{\dot{\alpha}}{\alpha} + \frac{\dot{\beta}}{\beta} + \frac{\dot{\gamma}}{\gamma} + \frac{\dot{\delta}}{\delta}\right) - \frac{\ddot{\alpha}}{\alpha} - \frac{\ddot{\beta}}{\beta} - \frac{\ddot{\gamma}}{\gamma} - \frac{\ddot{\delta}}{\delta} - \frac{1}{2}\frac{\dot{\varepsilon}^2}{\varepsilon^2}\right], \tag{5.2.4}$$

$$\frac{\tilde{a}^2}{\tilde{d}^2}\left[-\frac{\tilde{a}''}{\tilde{a}} - \frac{\tilde{a}'}{\tilde{a}}\left(3\frac{\tilde{a}'}{\tilde{a}} - \frac{\tilde{d}'}{\tilde{d}}\right) - \frac{1}{9}\alpha_0^2\frac{\delta^2}{\varepsilon^2}\frac{\tilde{d}^2}{\tilde{V}^2} - \frac{\sqrt{2}}{3}\alpha_0\frac{\delta}{\varepsilon}\frac{\tilde{d}}{\tilde{V}}\left(\delta(y) - \delta(y - \pi\lambda)\right)\right]$$
$$= \frac{\delta^2}{n^2}\left[\frac{\dot{\alpha}}{\alpha}\left(\frac{\dot{n}}{n} - \frac{\dot{\beta}}{\beta} - \frac{\dot{\gamma}}{\gamma} - \frac{\dot{\delta}}{\delta}\right) - \frac{\ddot{\alpha}}{\alpha} - \frac{n^2}{2\alpha^2\beta^2\gamma^2}\left(\left(\beta^2 - \gamma^2\right)^2 - \alpha^4\right)\right], \tag{5.2.5}$$

$$\frac{\tilde{a}^2}{\tilde{d}^2}\left[-\frac{\tilde{a}''}{\tilde{a}} - \frac{\tilde{a}'}{\tilde{a}}\left(3\frac{\tilde{a}'}{\tilde{a}} - \frac{\tilde{d}'}{\tilde{d}}\right) - \frac{1}{9}\alpha_0^2\frac{\delta^2}{\varepsilon^2}\frac{\tilde{d}^2}{\tilde{V}^2} - \frac{\sqrt{2}}{3}\alpha_0\frac{\delta}{\varepsilon}\frac{\tilde{d}}{\tilde{V}}\left(\delta(y) - \delta(y - \pi\lambda)\right)\right]$$
$$= \frac{\delta^2}{n^2}\left[\frac{\dot{\beta}}{\beta}\left(\frac{\dot{n}}{n} - \frac{\dot{\alpha}}{\alpha} - \frac{\dot{\gamma}}{\gamma} - \frac{\dot{\delta}}{\delta}\right) - \frac{\ddot{\beta}}{\beta} - \frac{n^2}{2\alpha^2\beta^2\gamma^2}\left(\left(\alpha^2 - \gamma^2\right)^2 - \beta^4\right)\right], \tag{5.2.6}$$

$$
\begin{aligned}
&\frac{\tilde{a}^2}{\tilde{d}^2}\left[-\frac{\tilde{a}''}{\tilde{a}}-\frac{\tilde{a}'}{\tilde{a}}\left(3\frac{\tilde{a}'}{\tilde{a}}-\frac{\tilde{d}'}{\tilde{d}}\right)-\frac{1}{9}\alpha_0^2\frac{\delta^2}{\varepsilon^2}\frac{\tilde{d}^2}{\tilde{V}^2}-\frac{\sqrt{2}}{3}\alpha_0\frac{\delta}{\varepsilon}\frac{\tilde{d}}{\tilde{V}}\left(\delta(y)-\delta(y-\pi\lambda)\right)\right]\\
=\ &\frac{\delta^2}{n^2}\left[\frac{\dot{\gamma}}{\gamma}\left(\frac{\dot{n}}{n}-\frac{\dot{\alpha}}{\alpha}-\frac{\dot{\beta}}{\beta}-\frac{\dot{\delta}}{\delta}\right)-\frac{\ddot{\gamma}}{\gamma}-\frac{n^2}{2\alpha^2\beta^2\gamma^2}\left(\left(\alpha^2-\beta^2\right)^2-\gamma^4\right)\right], \qquad (5.2.7)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\tilde{a}^2}{\tilde{d}^2}\left[4\left(\frac{\tilde{a}'}{\tilde{a}}\frac{\tilde{d}'}{\tilde{d}}-\frac{\tilde{a}''}{\tilde{a}}\right)-\frac{1}{9}\alpha_0^2\frac{\delta^2}{\varepsilon^2}\frac{\tilde{d}^2}{\tilde{V}^2}-\frac{1}{2}\frac{\tilde{V}'^2}{\tilde{V}^2}\right.\\
+\ &\left.\frac{4\sqrt{2}}{3}\alpha_0\frac{\delta}{\varepsilon}\frac{\tilde{d}}{\tilde{V}}\left(\delta(y)-\delta(y-\pi\lambda)\right)\right]\\
=\ &\frac{\delta^2}{n^2}\left[\frac{\dot{\delta}}{\delta}\left(\frac{\dot{n}}{n}-\frac{\dot{\alpha}}{\alpha}-\frac{\dot{\beta}}{\beta}-\frac{\dot{\gamma}}{\gamma}\right)-\frac{\ddot{\delta}}{\delta}\right]. \qquad (5.2.8)
\end{aligned}
$$

The equation of motion (5.1.10) for the scalar field V is

$$
\begin{aligned}
&\frac{\tilde{a}^2}{\tilde{d}^2}\left[4\frac{\tilde{a}'}{\tilde{a}}\frac{\tilde{V}'}{\tilde{V}}-\frac{\tilde{d}'}{\tilde{d}}\frac{\tilde{V}'}{\tilde{V}}-\frac{\tilde{V}''}{\tilde{V}}+\frac{\tilde{V}'^2}{\tilde{V}^2}-\frac{2}{3}\alpha_0^2\frac{\delta^2}{\varepsilon^2}\frac{\tilde{d}^2}{\tilde{V}^2}\right.\\
-\ &\left.2\sqrt{2}\alpha_0\frac{\delta}{\varepsilon}\frac{\tilde{d}}{\tilde{V}}\left(\delta(y)-\delta(y-\pi\lambda)\right)\right]\\
=\ &\frac{\delta^2}{n^2}\left[\frac{\dot{\varepsilon}}{\varepsilon}\left(\frac{\dot{\alpha}}{\alpha}+\frac{\dot{\beta}}{\beta}+\frac{\dot{\gamma}}{\gamma}+\frac{\dot{\delta}}{\delta}-\frac{\dot{n}}{n}\right)+\frac{\ddot{\varepsilon}}{\varepsilon}-\frac{\dot{\varepsilon}^2}{\varepsilon^2}\right]. \qquad (5.2.9)
\end{aligned}
$$

In fact, $\alpha(\tau), \beta(\tau), \gamma(\tau)$ are vorldvolume scale factors and $\delta(\tau)$ is an orbifold scale factor. It appears that the equations of motion (5.2.4)–(5.2.9) are fully separable into an orbifold-dependent part and a spacetime-dependent part provided [76, 159]

$$
n(\tau)=1, \qquad \delta(\tau)=\varepsilon(\tau), \qquad (5.2.10)
$$

where the first condition is simply the choice of the lapse function, while the second tells us that Calabi-Yau space is tracking the orbifold. One can show that orbifold-dependent part can be solved by

$$
\tilde{a} \ = \ a_0 H^{1/2}(y),
$$

$$\tilde{d} = d_0 H^2(y),$$
$$\tilde{V} = d_0 H^3(y), \qquad (5.2.11)$$

where

$$H(y) = \frac{\sqrt{2}}{3}\alpha_0 \mid y \mid + h_0, \qquad (5.2.12)$$

$$H''(y) = \frac{2\sqrt{2}}{3}\alpha_0 \left[\delta(y) - \delta(y - \pi\lambda)\right], \qquad (5.2.13)$$

and we have applied

$$\mid y \mid' = \epsilon(y) - \epsilon(y - \pi\lambda) - 1, \qquad (5.2.14)$$

so that

$$\mid y \mid'' = 2\delta(y) - 2\delta(y - \pi\lambda), \qquad (5.2.15)$$

(factor 2 comes from the fact that y is periodic) and

$$\epsilon(y) = 1 \quad if \quad y \geq 0, \qquad (5.2.16)$$

$$\epsilon(y) = -1 \quad if \quad y < 0. \qquad (5.2.17)$$

After all these substitutions one can write down 5-dimensional metric in the form

$$ds_5^2 = -a_0^2 H(y) d\tau^2 + a_0^2 H(y) \Big[\alpha^2(\tau)(\sigma^1)^2$$
$$+ \beta^2(\tau)(\sigma^2)^2 + \gamma^2(\tau, y)(\sigma^3)^2\Big] + d_0^2 H^4(y) \delta^2(\tau) dy^2. \qquad (5.2.18)$$

Elementary solutions of Hořava-Witten theory for Friedmann flat geometry analogous to (3.2.4) in pre-big-bang cosmology are given by taking one scale factor $\bar{a} \equiv \alpha = \beta = \gamma$ in (5.2.18) and read

$$\bar{a}(\tau) = \mid \tau \mid^{p_\mp}, \quad p_\mp = \frac{3}{11} \mp \frac{4}{11\sqrt{3}},$$

$$\delta(\tau) = \mid \tau \mid^{q_\mp}, \quad q_\pm = \frac{2}{11} \pm \frac{4\sqrt{3}}{11}, \qquad (5.2.19)$$

for the worldvolume and orbifold respectively. The meaning isotropic solutions (5.2.19) was discussed in [159] (note that numerically $p_+ = 0.48$, $p_- = 0.06$

and $q_- = -0.45$, $q_+ = 0.81$). In fact, there are two branches: each one for negative and positive values of time coordinate τ (negative values of time can be achieved by taking $-\tau$ instead of $+\tau$ in (5.3.1), or, simply by taking the modulus). If $\tau < 0$ one has $(-)$ branch and if $\tau > 0$ one has $(+)$ branch. For $(-)$ branch both the vorldvolume (of 3-dimensional space) and the orbifold contract for p_- and q_+ while the worldvolume contracts and the orbifold expands (superinflationary – cf. pre-big-bang scenario of Chapter 3) for p_+ and q_-. For $(+)$ branch the worldvolume and the orbifold expand for p_- and q_+ while the wordvolume expands and the orbifold contracts for p_+ and q_-.

After separation of the equations (5.2.4)–(5.2.9) and the choice of gauge (5.2.10) [159] we obtain the following set of time-dependent field equations (note that equations (5.2.7) and (5.2.9) become identical so we reduce the number of equations to five)

$$\frac{\ddot{\alpha}}{\alpha} + \frac{\ddot{\beta}}{\beta} + \frac{\ddot{\gamma}}{\gamma} + \frac{\ddot{\delta}}{\delta} = -\frac{1}{2}\frac{\dot{\delta}^2}{\delta^2}, \tag{5.2.20}$$

$$\frac{\ddot{\alpha}}{\alpha} + \frac{\dot{\alpha}}{\alpha}\left(\frac{\dot{\beta}}{\beta} + \frac{\dot{\gamma}}{\gamma} + \frac{\dot{\delta}}{\delta}\right) = \frac{1}{2\alpha^2\beta^2\gamma^2}\left[\left(\beta^2 - \gamma^2\right)^2 - \alpha^4\right], \tag{5.2.21}$$

$$\frac{\ddot{\beta}}{\beta} + \frac{\dot{\beta}}{\beta}\left(\frac{\dot{\alpha}}{\alpha} + \frac{\dot{\gamma}}{\gamma} + \frac{\dot{\delta}}{\delta}\right) = \frac{1}{2\alpha^2\beta^2\gamma^2}\left[\left(\alpha^2 - \gamma^2\right)^2 - \beta^4\right], \tag{5.2.22}$$

$$\frac{\ddot{\gamma}}{\gamma} + \frac{\dot{\gamma}}{\gamma}\left(\frac{\dot{\alpha}}{\alpha} + \frac{\dot{\beta}}{\beta} + \frac{\dot{\gamma}}{\gamma}\right) = \frac{1}{2\alpha^2\beta^2\gamma^2}\left[\left(\alpha^2 - \beta^2\right)^2 - \gamma^4\right], \tag{5.2.23}$$

$$\frac{\ddot{\delta}}{\delta} + \frac{\dot{\delta}}{\delta}\left(\frac{\dot{\alpha}}{\alpha} + \frac{\dot{\beta}}{\beta} + \frac{\dot{\gamma}}{\gamma}\right) = 0, \tag{5.2.24}$$

which, except for the right-hand side of (5.2.20), is the same set as the set of equations (3.4.57)–(3.4.58) of Section 3.4 for the common sector of superstring cosmology in the string frame, provided that we take dilaton field ϕ to be equal to $(-\ln\delta)$ and also neglect the axion (i.e. take $A = 0$).

A new time coordinate is introduced to simplify the field equations by (compare (3.4.62))

$$d\eta = \frac{d\tau}{\alpha\beta\gamma\delta}. \tag{5.2.25}$$

From now on we will use the notation $(...)_{,\eta} = d/d\eta$. To further simplify the equations we additionally define

$$\tilde{\alpha} = \ln\alpha \qquad \tilde{\beta} = \ln\beta \qquad \tilde{\gamma} = \ln\gamma \qquad \tilde{\delta} = \ln\delta, \tag{5.2.26}$$

so that the set of equations (5.2.20)–(5.2.24) reads as

$$\left(\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} + \tilde{\delta}\right)_{,\eta\eta} + \frac{1}{2}\tilde{\delta}^2_{,\eta} = 2\left(\tilde{\alpha}_{,\eta}\tilde{\beta}_{,\eta} + \tilde{\alpha}_{,\eta}\tilde{\gamma}_{,\eta} + \tilde{\beta}_{,\eta}\tilde{\gamma}_{,\eta}\right)$$

$$+2\left(\tilde{\alpha}_{,\eta} + \tilde{\beta}_{,\eta} + \tilde{\gamma}_{,\eta}\right)\tilde{\delta},\eta, \tag{5.2.27}$$

$$2\tilde{\alpha}_{,\eta\eta} = \left[\left(\beta^2 - \gamma^2\right)^2 - \alpha^4\right]\delta^2, \tag{5.2.28}$$

$$2\tilde{\beta}_{,\eta\eta} = \left[\left(\alpha^2 - \gamma^2\right)^2 - \beta^4\right]\delta^2, \tag{5.2.29}$$

$$2\tilde{\gamma}_{,\eta\eta} = \left[\left(\beta^2 - \gamma^2\right)^2 - \alpha^4\right]\delta^2, \tag{5.2.30}$$

$$\tilde{\delta}_{\prime\eta\eta} = 0. \tag{5.2.31}$$

These equations are the same as the pre-big-bang cosmology Mixmaster equations in string frame (3.4.69)–(3.4.72) if we take $\tilde{\delta} = -\phi = -M\eta + \text{const.}$, or, as the 5-dimensional vacuum Mixmaster equations (31)–(32) of Halpern [114].

Now we consider suitable initial conditions expressed in terms of the Kasner parameters and discuss the general behaviour of Bianchi type IX Hořava-Witten cosmology on the approach to singularity.

5.3 Kasner type of Hořava-Witten cosmologies

The Kasner solutions are obtained as approximate solutions of the equations (5.2.20)–(5.2.24) when the right-hand sides (describing the curvature anisotropies) are neglected. In terms of τ-time, they are [75]

$$\alpha = \alpha_0\tau^{p_1},$$

$$\begin{aligned} \beta &= \beta_0 \tau^{p_2}, \\ \gamma &= \gamma_0 \tau^{p_3}, \\ \delta &= \delta_0 \tau^{p_4}, \end{aligned} \tag{5.3.1}$$

while

$$\tilde{\delta} = -\ln \delta_0 - p_4 \ln \tau. \tag{5.3.2}$$

From (5.2.27)–(5.2.31) we have the following algebraic conditions for the Kasner indices, p_i :

$$p_1 + p_2 + p_3 + p_4 = 1, \tag{5.3.3}$$

and

$$p_1^2 + p_2^2 + p_3^2 + \frac{3}{2} p_4^2 = 1. \tag{5.3.4}$$

This, in particular, proves that the isotropic Friedmann case (5.2.19) given by Kasner indices equal

$$p_1 = p_2 = p_3 = p_{\mp} = \frac{3}{11} \mp \frac{4}{11\sqrt{3}}, \tag{5.3.5}$$

$$p_4 = q_{\pm} = \frac{2}{11} \pm \frac{4\sqrt{3}}{11} \tag{5.3.6}$$

is included.

Notice that the conditions (5.3.3)–(5.3.4) are different from the conditions which emerge in pre-big-bang cosmology where the role of the fifth coordinate is played by the dilaton (see (3.4.100)–(3.4.101) of Section 3.3). They are also different from Mixmaster Kaluza-Klein five-dimensional models where the homogeneity group acts on four-dimensional hypersurfaces of constant time (see Eq. (33) of [114]). The reason for that is simply the fact that the fifth coordinate in Hořava-Witten cosmology is an orbifold. The isotropization of the models under consideration means that the Kasner indices reach the values (5.3.5)–(5.3.6). Finally, the isotropization of 5-dimensional Kaluza-Klein models in supergravity as first considered by Chodos and Detweiler [51] would require the different values of the Kasner indices, namely $p_1 = p_2 = p_3 = -p_4 = 1/2$ which fulfill the conditions (5.3.3) and (5.3.4) without a factor 3/2 in front of p_4 in (5.3.4).

Having given the conditions (5.3.3)–(5.3.4), one can express the indices p_2 and p_3 by using p_1 and p_4, i.e.,

$$p_2 = \frac{1}{2}\left[(1 - p_1 - p_4) - \sqrt{-3p_1^2 + 2p_1(1 - p_4) + 1 + 2p_4(1 - 2p_4)}\right],$$

$$p_3 = \frac{1}{2}\left[(1 - p_1 - p_4) + \sqrt{-3p_1^2 + 2p_1(1 - p_4) + 1 + 2p_4(1 - 2p_4)}\right]. \quad (5.3.7)$$

Since the expression under the square root in (5.3.7) should be nonnegative, we obtain the restriction on the permissible values of p_4,

$$q_- \le p_4 \le q_+. \quad (5.3.8)$$

Some particular choices are of interest. If one takes $p_4 = 0$ one recovers vacuum general relativity limit with Kasner indices $-1/3 \le p_1 \le 0, 0 \le p_2 \le 2/3, 2/3 \le p_3 \le 1$. The range dividing case is for $p_4 = 2/11$ with the following ordering of the Kasner indices

$$\frac{3}{11} - \frac{4}{11\sqrt{3}}\sqrt{11} \le p_1 \le \frac{3}{11} - \frac{2}{11\sqrt{3}}\sqrt{11},$$

$$\frac{3}{11} - \frac{2}{11\sqrt{3}}\sqrt{11} \le p_2 \le \frac{3}{11} + \frac{2}{11\sqrt{3}}\sqrt{11}, \quad (5.3.9)$$

$$\frac{3}{11} + \frac{2}{11\sqrt{3}}\sqrt{11} \le p_3 \le \frac{3}{11} + \frac{4}{11\sqrt{3}}\sqrt{11}.$$

However, we are interested in knowing whether the curvature terms on the right-hand side of the field equations (5.2.28)–(5.2.30) really increase as $\eta \to -\infty$ ($\tau \to 0$ – approach to singularity for (+) branch) since from (5.2.25) and (5.3.1) we get

$$\eta = \eta_0 + \ln \tau, \quad (5.3.10)$$

and $\eta_0 = \text{const}$. This would require either $\alpha^4\delta^2, \beta^4\delta^2$, or $\gamma^4\delta^2$ to increase if the transition to another Kasner epoch is to occur [114, 123]. Since

$$\alpha^4\delta^2 \propto \tau^{(2p_1+p_4)} = \tau^{(1+p_1-p_2-p_3)},$$

$$\beta^4\delta^2 \propto \tau^{(2p_2+p_4)} = \tau^{(1+p_2-p_3-p_1)}, \quad (5.3.11)$$

$$\gamma^4\delta^2 \propto \tau^{(2p_3+p_4)} = \tau^{(1+p_3-p_1-p_2)},$$

we need one of the following three conditions to be fulfilled

$$
\begin{aligned}
2p_1 + p_4 &= 1 + p_1 - p_2 - p_3 < 0, \\
2p_2 + p_4 &= 1 + p_2 - p_3 - p_1 < 0, \\
2p_3 + p_4 &= 1 + p_3 - p_1 - p_2 < 0.
\end{aligned} \tag{5.3.12}
$$

From Kasner conditions (5.3.3)–(5.3.4) we are free to choose only two parameters so that we can write

$$
\begin{aligned}
p_4 &= 1 - p_1 - p_2 - p_3, \\
p_3 &= \frac{3}{5}(1 - p_1 - p_2) \\
&\pm \frac{2}{5}\left[-4\left(p_1^2 + p_2^2\right) - 3p_1p_2 + 3p_1 + 3p_2 + 1\right]^{\frac{1}{2}},
\end{aligned} \tag{5.3.13}
$$

which gives also the condition for p_3 to be real:

$$
1 - 4\left(p_1^2 + p_2^2\right) - 3p_1p_2 + 3p_1 + 3p_2 \geq 0. \tag{5.3.14}
$$

In (5.3.13) we take the $+$ sign for $p_4 < 2/11$ and the $-$ sign for $p_4 > 2/11$. The conditions (5.3.12) become (compare [114])

$$
\begin{aligned}
p_1^2 + 4p_2^2 - p_1p_2 + p_2 - p_1 &< 0, \\
4p_1^2 + p_2^2 - p_1p_2 + p_1 - p_2 &< 0, \\
4\left(p_1^2 + p_2^2\right) + 7p_1p_2 - 7(p_1 + p_2) + 3 &> 0.
\end{aligned} \tag{5.3.15}
$$

One should remind here that for ($-$) branch we have to take $(-\tau)$ in (5.3.1), (5.3.2) and (5.3.11) which leads to the same conditions (5.3.15).

The plot of the conditions (5.3.14)–(5.3.15) is given in Fig. 5.3. The chaotic oscillations (Kasner-to-Kasner transitions) can start in any region except the narrow range surrounding the isotropic points $p_1 = p_2 = p_+ (q = q_-)$ and $p_1 = p_2 = p_- (q = q_+)$. However, such chaotic oscillations would continue indefinitely provided there were no such regions at all (this is the case of vacuum general relativity, for example). Here, once the Kasner parameters fall into the region surrounding the isotropic Friedmann p_+ or p_- solutions of

[159], the chaotic oscillations cease, so that there is *no chaos* in such Hořava-Witten cosmologies.

The most general inhomogeneous solutions of (5.2.4)–(5.2.9) have been studied qualitatively in [151].

In fact, there is a relationship between the pre-big-bang (3.1.9) and Hořava-Witten (5.1.8) solutions and the result of our previous section is an example of generation of solutions from those known in one theory, into the other – in particular, into solutions of Hořava-Witten theory [151]. This is possible due to the conformal relation between the theories:

$$g^{E}_{\mu\nu} = e^{-\phi} g^{S}_{\mu\nu} = e^{\frac{1}{2}\phi} g^{HW}_{\mu\nu}, \tag{5.3.16}$$

where E, S, HW refer to Einstein frame, string frame and Hořava-Witten, respectively, and the relation is true, provided 3-branes are given by the separable ansatz equations (5.2.3). In particular, lots of exact solutions are available for the Einstein frame (general relativity + scalar field/stiff fluid matter) – having them, one is able to generate Hořava-Witten type solutions and study their properties.

Our conclusions are as follows. Hořava-Witten theory admits Mixmaster type cosmology with Kasner type asymptotic solutions of the Bianchi type I class. Due to conformal relations one can generate Hořava-Witten cosmological solutions (with separable 3-brane 5.2.3) of many types and study their properties. Hořava-Witten Mixmaster cosmology (for truncated spectrum of particles), similarly as pre-big-bang (truncated) cosmology does not admit chaos.

Finally, one should mention some open issues in the topic. Firstly, one could study non-separable ansätze for Hořava-Witten cosmology and investigate how they relate to pre-big-bang. Secondly, one could study Mixmaster behaviour of cosmological models which involve non-truncated spectrum of particles and inhomogeneities [69, 70].

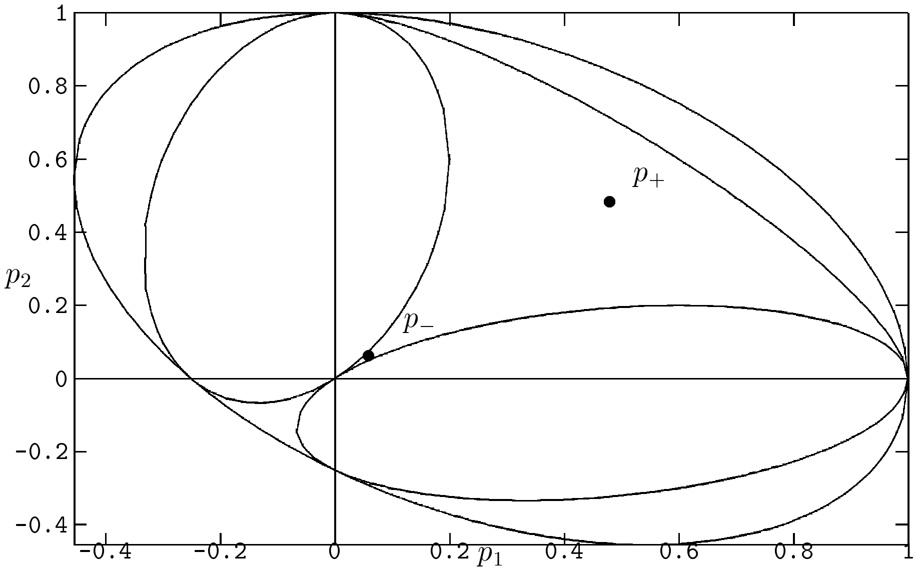

Figure 5.3: The range of Kasner indices p_1 and p_2 which fulfill the conditions (5.3.14)–(5.3.15). The appearance of the isotropic Friedmann cases at $p_1 = p_2 = p_-$ and $p_1 = p_2 = p_+$ prevents chaotic oscillations in the shaded region that surrounds them. For the values of Kasner indices p_1 and p_2 from that region chaotic oscillations are not possible to begin. On the other hand, even if after some number of oscillations from one Kasner epoch to the other, the values of p_1 and p_2 will fall into that region, the chaotic oscillations of the scale factors stop. This reflects nonchaotic behaviour of such Bianchi IX Hořava-Witten cosmologies

Appendix A

Ricci tensor for Bianchi IX universes

In this Appendix give the components of Ricci tensor for the axisymmetric Bianchi IX model in terms of coordinate frames rather than in the orthonormal frames of Section 3.4. For the sake of completness, we start with a general metric.

The metric (3.4.1) in a coordinate frame has the following components

$$g_{\bar{0}\bar{0}} = 1, \tag{A.1}$$

$$g_{\bar{1}\bar{1}} = -c^2(t), \tag{A.2}$$

$$g_{\bar{2}\bar{2}} = -\Big[a^2(t)\cos^2\psi + b^2(t)\sin^2\psi\Big], \tag{A.3}$$

$$g_{\bar{3}\bar{3}} = -\sin^2\theta\Big[a^2(t)\sin^2\psi + b^2(t)\cos^2\psi\Big] - c^2(t)\cos^2\theta, \tag{A.4}$$

$$g_{\bar{1}\bar{3}} = -c^2(t)\cos\theta, \tag{A.5}$$

$$g_{\bar{2}\bar{3}} = -\sin\psi\cos\psi\sin\theta\Big[a^2(t) - b^2(t)\Big]. \tag{A.6}$$

The Ricci tensor components in the coordinate frame can be calculated using the relations [30] (cf. (3.4.1)–(3.4.4))

$$R_{\alpha\beta} = e^j_\alpha e^k_\beta R_{jk}, \tag{A.7}$$

$$R^\beta_\alpha = e^j_\alpha e^\beta_k R^k_j, \tag{A.8}$$

where $R_{\alpha\beta}$ is the Ricci tensor in the coordinate frame while R_{jk} is the Ricci tensor in the orthonormal frame (correspondingly $\alpha, \beta = \bar{0}, \bar{1}, \bar{2}, \bar{3}$ are the coordinate frame indices and $i, j = 0, 1, 2, 3$ are the orthonormal frame indices). Thus, $1, 2, 3$ refer to $\sigma^1, \sigma^2, \sigma^3$, while $\bar{1}, \bar{2}, \bar{3}$ refer to ψ, θ, φ respectively. According to (3.4.2)–(3.4.4),

$$\begin{aligned}
e^1_{\bar{2}} &= a\cos\psi, \\
e^1_{\bar{3}} &= a\sin\psi\sin\theta, \\
e^2_{\bar{2}} &= b\sin\psi, \\
e^2_{\bar{3}} &= -b\cos\psi\sin\theta, \\
e^3_{\bar{1}} &= c, \\
e^3_{\bar{3}} &= c\cos\theta,
\end{aligned} \tag{A.9}$$

and

$$e^{\bar{1}}_1 = -\frac{1}{a}\sin\psi\cot\theta, \tag{A.10}$$

$$\begin{aligned}
e^{\bar{1}}_2 &= \frac{1}{b}\cos\psi\cot\theta, \\
e^{\bar{1}}_3 &= \frac{1}{c}, \\
e^{\bar{2}}_1 &= \frac{1}{a}\cos\psi, \\
e^{\bar{2}}_2 &= \frac{1}{b}\sin\psi, \\
e^{\bar{3}}_1 &= \frac{1}{a}\frac{\sin\psi}{\sin\theta}, \\
e^{\bar{3}}_2 &= -\frac{1}{b}\frac{\cos\psi}{\sin\theta}.
\end{aligned} \tag{A.11}$$

Then, the Ricci tensor components in the coordinate frame are given by

$$R_{\bar{1}\bar{1}} = c^2 R_{33}, \tag{A.12}$$

$$R_{\bar{2}\bar{2}} = a^2\cos^2\psi R_{11} + b^2\sin^2\psi R_{22}, \tag{A.13}$$

$$R_{\bar{3}\bar{3}} = a^2\sin^2\psi\sin^2\theta R_{11} + b^2\cos^2\psi\sin^2\theta R_{22} + c^2\cos^2\theta R_{33}, \tag{A.14}$$

$$R_{\bar{1}\bar{3}} = c^2\cos\theta R_{33}, \tag{A.15}$$

$$R_{\bar{2}\bar{3}} = \sin\psi\cos\psi\sin\theta\left(a^2 R_{11} - b^2 R_{22}\right), \tag{A.16}$$

$$R_{\bar{1}\bar{2}} = 0, \tag{A.17}$$

and

$$R^{\bar{1}}_{\bar{1}} = R^3_3, \tag{A.18}$$

$$R^{\bar{2}}_{\bar{2}} = \cos^2\psi R^1_1 + \sin^2\psi R^2_2, \tag{A.19}$$

$$R^{\bar{3}}_{\bar{3}} = \sin^2\psi R^1_1 + \cos^2\psi R^2_2, \tag{A.20}$$

$$R^{\bar{1}}_{\bar{3}} = \left[-\sin^2\psi R^1_1 - \cos^2\psi R^2_2 + R^3_3\right]\cos\theta, \tag{A.21}$$

$$R^{\bar{2}}_{\bar{3}} = \sin\psi\cos\psi\sin\theta\left(R^1_1 - R^2_2\right), \tag{A.22}$$

$$R^{\bar{3}}_{\bar{2}} = \frac{\sin\psi\cos\psi}{\sin\theta}\left(R^1_1 - R^2_2\right), \tag{A.23}$$

$$R^{\bar{1}}_{\bar{2}} = -\sin\psi\cos\psi\cot\theta\left(R^1_1 - R^2_2\right), \tag{A.24}$$

$$R^{\bar{3}}_{\bar{1}} = 0, \tag{A.25}$$

$$R^{\bar{2}}_{\bar{1}} = 0. \tag{A.26}$$

If two axes are the same ($a(t) = b(t)$) the metric (3.4.1) (or its components given by (A.1)–(A.6)) simplifies to the form given by (see 2.3.99) [35, 10]

$$ds^2 = dt^2 - c^2\left(d\psi + \cos\theta d\varphi\right)^2 - a^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{A.27}$$

The nonvanishing Christoffel symbols for the metric (A.27) are

$$\Gamma^{\bar{1}}_{\bar{0}\bar{1}} = \frac{\dot{c}}{c}, \quad \Gamma^{\bar{2}}_{\bar{0}\bar{2}} = \frac{\dot{a}}{a}, \quad \Gamma^{\bar{3}}_{\bar{0}\bar{3}} = \frac{\dot{a}}{a},$$

$$\Gamma^{\bar{1}}_{\bar{0}\bar{3}} = \left(\frac{\dot{c}}{c} - \frac{\dot{a}}{a}\right)\cos\theta, \quad \Gamma^{\bar{0}}_{\bar{1}\bar{1}} = \dot{c}c, \quad \Gamma^{\bar{0}}_{\bar{2}\bar{2}} = \dot{a}a,$$

$$\Gamma^{\bar{0}}_{\bar{3}\bar{3}} = \dot{c}c\cos^2\theta + \dot{a}a\sin^2\theta, \quad \Gamma^{\bar{0}}_{\bar{1}\bar{3}} = \dot{c}c\cos\theta,$$

$$\Gamma^{\bar{1}}_{\bar{2}\bar{1}} = \frac{1}{2}\frac{c^2}{a^2}\cot\theta, \quad \Gamma^{\bar{1}}_{\bar{2}\bar{3}} = \frac{1}{2\sin\theta}\left(\frac{c^2 - a^2}{a^2}\cos^2\theta - 1\right), \tag{A.28}$$

$$\Gamma^{\bar{2}}_{\bar{3}\bar{1}} = \frac{1}{2}\frac{c^2}{a^2}\sin\theta, \quad \Gamma^{\bar{2}}_{\bar{3}\bar{3}} = \sin\theta\cos\theta\frac{c^2 - a^2}{a^2},$$

$$\Gamma^{\bar{3}}_{\bar{1}\bar{2}} = -\frac{1}{2}\frac{c^2}{a^2}\frac{1}{\sin\theta}, \qquad \Gamma^{\bar{3}}_{\bar{2}\bar{3}} = \cot\theta\left(1 - \frac{1}{2}\frac{c^2}{a^2}\right).$$

The gradients of the dilaton calculated with respect to the metric (A.27) are given by

$$\nabla_{\bar{0}}\nabla^{\bar{0}}\phi = \ddot{\phi}, \tag{A.29}$$

$$\nabla_{\bar{1}}\nabla^{\bar{1}}\phi = \frac{\dot{c}}{c}\dot{\phi}, \tag{A.30}$$

$$\nabla_{\bar{2}}\nabla^{\bar{2}}\phi = \nabla_{\bar{3}}\nabla^{\bar{3}} = \frac{\dot{a}}{a}\dot{\phi}, \tag{A.31}$$

$$\nabla_{\bar{3}}\nabla^{\bar{1}}\phi = \dot{\phi}\left(\frac{\dot{c}}{c} - \frac{\dot{a}}{a}\right)\cos\theta, \tag{A.32}$$

and

$$\nabla_{\bar{0}}\phi\nabla^{\bar{0}}\phi = \dot{\phi}^2. \tag{A.33}$$

The nonzero components of the Ricci tensor in a coordinate frame for the metric (A.27) can be obtained from (A.12)–(A.26) by putting $R^1_1 = R^2_2$ and using (3.4.9)–(3.4.12) for $a = b$ or directly using the Christoffel symbols. This gives

$$\begin{gathered} R_{\bar{1}\bar{1}} = \ddot{c}c + 2\dot{c}c\frac{\dot{a}}{a} + \frac{1}{2}\frac{c^4}{a^4}, \qquad R_{\bar{1}\bar{3}} = R_{\bar{1}\bar{1}}\cos\theta, \\ R_{\bar{2}\bar{2}} = \ddot{a}a + \dot{a}^2 + \dot{a}a\frac{\dot{c}}{c} + 1 - \frac{1}{2}\frac{c^2}{a^2}, \qquad R_{\bar{0}\bar{0}} = -\frac{\ddot{c}}{c} - 2\frac{\ddot{a}}{a}, \\ R_{\bar{3}\bar{3}} = R_{\bar{1}\bar{1}}\cos^2\theta + R_{\bar{2}\bar{2}}\sin^2\theta, \end{gathered} \tag{A.34}$$

hence

$$-R^{\bar{0}}_{\bar{0}} = \frac{\ddot{c}}{c} + 2\frac{\ddot{a}}{a}, \tag{A.35}$$

$$-R^{\bar{1}}_{\bar{1}} = \frac{\ddot{c}}{c} + 2\frac{\dot{a}}{a}\frac{\dot{c}}{c} + \frac{1}{2}\frac{c^2}{a^4}, \tag{A.36}$$

$$-R^{\bar{2}}_{\bar{2}} = -R^{\bar{3}}_{\bar{3}} = \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\dot{a}}{a}\frac{\dot{c}}{c} + \frac{1}{a^2} - \frac{1}{2}\frac{c^2}{a^4}, \tag{A.37}$$

$$R^{\bar{1}}_{\bar{3}} = \left(\frac{\ddot{a}}{a} - \frac{\ddot{c}}{c} - \frac{\dot{a}}{a}\frac{\dot{c}}{c} + \frac{\dot{a}^2}{a^2} + \frac{1}{a^2} - \frac{c^2}{a^4}\right)\cos\theta, \tag{A.38}$$

and the Ricci scalar reads

$$R = -2\frac{\ddot{c}}{c} - 4\frac{\ddot{a}}{a} - 4\frac{\dot{a}}{a}\frac{\dot{c}}{c} - 2\frac{\dot{a}^2}{a^2} - 2\frac{1}{a^2} + \frac{1}{2}\frac{c^2}{a^4}. \tag{A.39}$$

Note that in the coordinate frame $R^{\bar{2}}_{\bar{2}} = R^{\bar{3}}_{\bar{3}}$, while in the orthonormal frame (cf. (3.4.10)–(3.4.11)) $R^1_1 = R^2_2$. This is reasonable, since the metric tensor is given by (A.27) for the former case, and by (3.4.1) with $a = b$ for the latter case, where the indices refer to the orthonormal basis rather than to the chosen coordinates.

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Kosmologie strunowe

Streszczenie

Monografia poświęcona jest zastosowaniu teorii superstrunowych w kosmologii. Teorie superstrunowe jako teorie unifikacji wszystkich czterech oddziaływań fundamentalnych włącznie z grawitacją, są niewątpliwie najlepszym sposobem opisu zjawisk zachodzących we wczesnych etapach ewolucji Wszechświata.

W pracy przedstawiono ewolucję strun próbnych w zakrzywionych czasoprzestrzeniach czarnych dziur oraz kosmologicznych. Szczególną uwagę poświęcono tutaj problemowi ewolucji strun zerowych jako zerowego przybliżenia strun z niezerowym napięciem. Ewolucja strun z niezerowym napięciem może mieć w ogólności charakter chaotyczny.

Głównym zagadnieniem rozważanym w niniejszej pracy jest ewolucja substratu strun w zakrzywionych czasoprzestrzeniach. Substrat ten składa się z trzech podstawowych modów strunowych: dylatonu, grawitonu i aksjonu, które determinują wielkoskalową ewolucję Wszechświata. Modem unifikującym pozostałe oddziaływania z grawitacją jest dylaton – pole skalarne typu Bransa-Dicke'a. Jego obecność jest niezbędna dla uzyskania podstawowych efektów charakterystycznych dla kosmologii strunowych, np. superinflacji, która zachodzi dla ujemnych czasów i usprawiedliwia nazwę rozważanych modeli kosmologicznych – kosmologie przed-Wielkim-Wybuchem.

W dalszej części monografii rozważa się jeszcze ogólniejsze modele kosmologiczne oparte na tzw. M-teorii – teorii zawierającej wszystkie teorie superstrunowe, opisywanej w jedenastu wymiarach czasoprzestrzennych. Takie modele znane są pod nazwą kosmologii membranowych (Hořavy-Wittena i Randall-Sundrum).