Cosmic singularities and varying constants

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1. Introduction.

Standard Einstein-Friedmann equations are two equations for three unknown functions of time $a(t), p(t), \varrho(t)$

$$\varrho = 3\left(\frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) , \tag{1}$$

$$p = -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) . \tag{2}$$

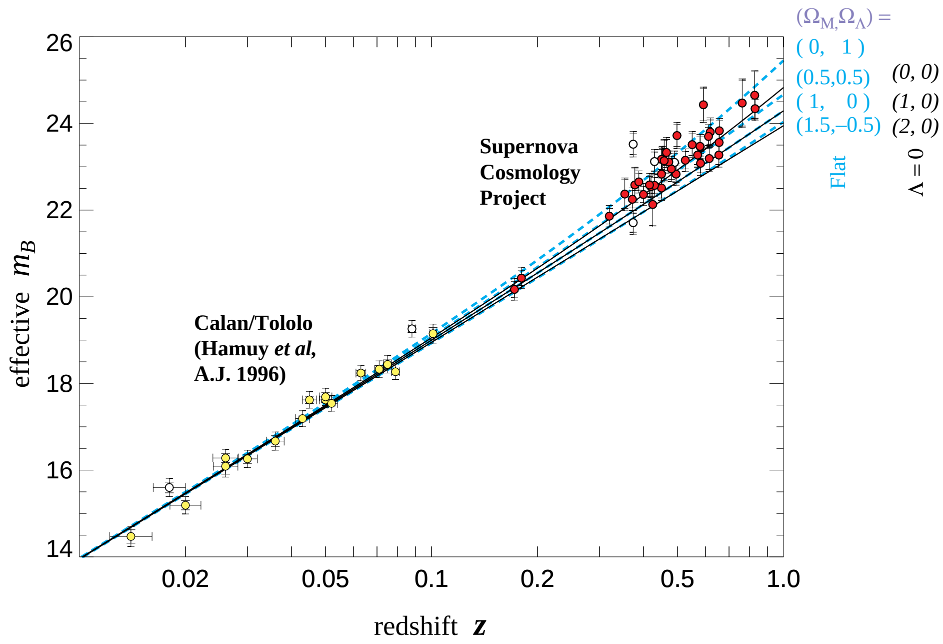
plus an equation of state, e.g., of a barotropic type ($w =$ const. ≥ -1):

$$p(t) = w\varrho(t) \qquad \to a(t) \propto t^{\frac{2}{3(w+1)}} . \tag{3}$$

Until very recently (including first supernovae results) most of cosmologists studied only simplest - say "standard" solutions - each of them starts with Big-Bang singularity in which $a \to 0$, $\varrho, p \to \infty$
– one of them (of $K = +1$) terminates at the second singularity (Big-Crunch) where $a \to 0$, $\varrho, p \to \infty$
– the other two ($K = 0, -1$) continue to an asymptotic emptiness $\varrho, p \to 0$ for $a \to \infty$.

BB and BC exhibit geodesic incompleteness and curvature blow-up.

However, first supernovae observations ...



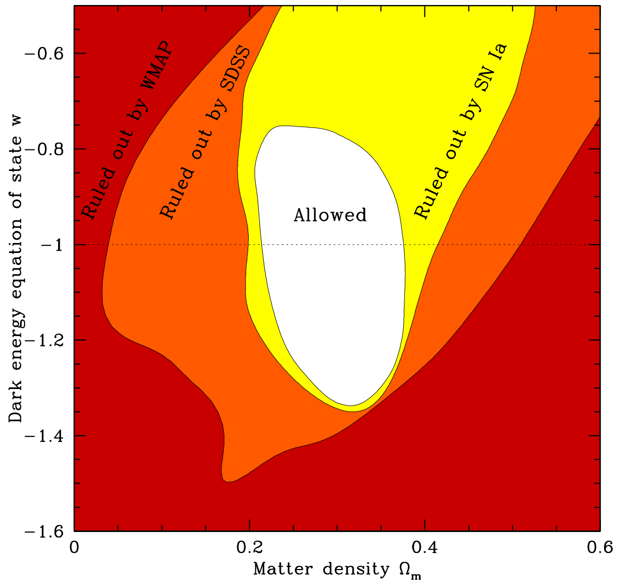
... gave evidence for the strong energy condition

$$\varrho + 3p \geq 0, \qquad \varrho + p \geq 0\ . \tag{4}$$

violation, but **the paradigm of the "standard" Big-Bang/Crunch singularities remained untouched.**

2. Standard and exotic singularities in cosmology.

WMAP + SDSS + Supernovae **combined bound on the dark energy barotropic index** w (Tegmark et al. (2004)):



- **showed that there was no sharp cut-off of the data at $p = -\varrho$!!!** so that
- **the dark energy with $p < -\varrho$ (phantom) can be admitted!**

More recent data:

- Knop et al. 2003 (from SNe + CMB + 2dFGRS combined) – $w = -1.05^{+0.15}_{-0.20}$ (statistical) ± 0.09 (systematic)
- Riess et al. 2004 ($w < -1$)
- Seljak et al. astro-ph/0604335 – $w = -1.04 \pm 0.06$
- though more recently Kowalski et al. (arXiv:0804.4142) analyzed 307 supernovae (Sne + BAO + CMB) – $w = -1.001^{+0.059}_{-0.063}$ (statistical) $^{+0.063}_{-0.066}$ (systematic); Amanullah et al. Ap.J. **716**, 712 (2010).

gave some evidence for possible cosmic "no-hair" theorem violation - **even a small fraction of phantom dark energy may dominate the evolution**

N(ull) E(nergy) C(ondition) $\varrho + p \geq 0$,

W(eak) E(nergy) C(ondition) $\varrho + p \geq 0, \rho \geq 0$,

D(ominant) E(nergy) C(ondition) $\mid p \mid \leq \varrho, \varrho \geq 0$ are violated!!!

Big-Rip (type I) as an exotic (neither BB nor BC) singularity.

Since for phantom $w < -1$, then for convenience we may take

$$| w + 1 | = -(w + 1) > 0 \,, \tag{5}$$

so $a(t) = t^{-2/3|w+1|}$ and the conservation law for phantom gives

$$\varrho \propto a^{3|w+1|} \,. \tag{6}$$

- Conclusion: the bigger the universe grows, the denser it is, and **it becomes dominated by phantom (which overcomes Λ-term)** – an exotic future singularity appears – Big-Rip $\varrho, p \to \infty$ for $a \to \infty$
- Curvature invariants R^2, $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ diverge at Big-Rip
- In a Big-Rip scenario everything is pulled apart on the approach to a Big-Rip in a reverse order (Caldwell et al. PRL '03). Specifically, for $w = -3/2$ Big-Rip will happen in 20 Gyr.

Sudden Future Singularity (type II) as an exotic singularity.

Observational support for a Big-Rip gave a push to studies some other exotic types of singularities as possible sources of dark energy.

Barrow (2004) proposed a Sudden Future Singularity (SFS) (or type II - Nojiri, Odintsov, Tsujikawa 2005) which:

– manifests as a singularity of pressure (or $\ddot{a}$) only

– leads to the dominant energy condition violation only and it emerges due to

a drop of the assumption about the imposition of an equation of state

$$p \neq p(\varrho), \qquad \text{no analytic form of this relation is given} \tag{7}$$

Only the form of the scale factor is given in the field equations:

$$a(t) = a_s \left[\delta + (1-\delta)\, y^m - \delta\, (1-y)^n\right] , \qquad y \equiv \frac{t}{t_s} \tag{8}$$

where $a_s \equiv a(t_s) = \text{const.}$ and $\delta, m, n = \text{const.}$

Apart from a Big-Bang at $t = 0$ there is a new type of singularity at $t = t_s$.

$$\dot{a} = a_s \left[\frac{m}{t_s} (1-\delta) y^{m-1} + \delta \frac{n}{t_s} (1-y)^{n-1} \right] , \quad (9)$$

$$\ddot{a} = \frac{a_s}{t_s^2} \left[m(m-1)(1-\delta) y^{m-2} - \delta n (n-1)(1-y)^{n-2} \right] . \quad (10)$$

Provided

$$1 < n < 2, \quad (11)$$

and using Einstein equations we get the following properties:

$$a = \text{const.}, \qquad \dot{a} = \text{const.} \qquad \varrho = \text{const.}$$
$$\ddot{a} \to -\infty \qquad p \to \infty \qquad \text{for} \qquad t \to t_s \quad (12)$$

Friedmann limit is easily obtained by taking the "nonstandardicity" parameter $\delta \to 0$.

Generalized Sudden Future singularities (type IIg).

Sudden future singularities may be generalized to GSFS if we take a general scale factor time derivative of an order r:

$$a^{(r)} = a_s \left[\frac{m(m-1)...(m-r+1)}{t_s^r} (1-\delta)\, y^{m-r} + (-1)^{r-1} \delta \frac{n(n-1)...(n-r+1)}{t_s^r} (1-y)^{n-r} \right] , \qquad (13)$$

and choose (Barrow 2004, Lake 2004) $r-1 < n < r$. Then for any integer r we have a singularity in the scale factor derivative $a^{(r)}$, and consequently in the appropriate pressure derivative $p^{(r-2)}$.

None of the energy conditions are violated for $r \geq 3$!!!

Finite Scale Factor (type III), Big Separation (type IV) and w-singularities (type V).

Type III singularities which we will call Finite Scale Factor - FSF singularities are characterized by the following conditions:

$$a = a_s = \text{const.},\ \varrho, \dot{a}_s \to \infty,\ |p|, \ddot{a}_s \to \infty$$

The simplest way to get them is to apply the scale factor as given previously for SFS, i.e.,

$$a(t) = a_s \left[\delta + (1-\delta)\, y^m - \delta\, (1-y)^n\right] , \qquad y \equiv \frac{t}{t_s} \tag{14}$$

where $a_f \equiv a(t_f) = \text{const.}$ and $\delta, A, m, n = \text{const.}$, but with the range of parameter n changed from $1 < n < 2$ onto

$$0 < n < 1$$

Big Separation - BS (type IV)

Type IV singularity is when:

$$a = a_s = \text{const.},\ \varrho \to 0,\ p \to 0,\ \dddot{a},\ \ddot{H} \to \infty \text{ etc.}$$

and so it is similar to Generalized Sudden Future singularity with only one exception: it also gives the divergence of the barotropic index in the barotropic equation of state

$$p(t) = w(t)\varrho(t)$$

i.e.,

$$w(t) \to \infty$$

Barotropic index $w-$singularity

Another exotic is a $w-$singularity only (without the divergence of the higher-derivatives of the scale factor). (Strangely, it really appears in physical theories such as $f(R)$ gravity (Starobinsky '80), scalar field gravity (Setare, Saridakis '09, and brane gravity Sahni, Shtanov '05)). We choose

$$a(t) = A + B\left(\frac{t}{t_s}\right)^{\frac{2}{3\gamma}} + C\left(D - \frac{t}{t_s}\right)^{n} , \tag{15}$$

where A, B, C, D, γ, n, and t_s are constants and impose the conditions:

$$a(0) = 0,\ a(t_s) = const. \equiv a_s,\ \dot{a}(t_s) = 0,\ \ddot{a}(t_s) = 0 , \tag{16}$$

which finally leads to the following form of the scale factor:

$w-$singularity

$$
\begin{aligned}
a(t) &= \frac{a_s}{1-\frac{3\gamma}{2}\left(\frac{n-1}{n-\frac{2}{3\gamma}}\right)^{n-1}} \\
&+ \frac{1-\frac{2}{3\gamma}}{n-\frac{2}{3\gamma}}\frac{n a_s}{1-\frac{2}{3\gamma}\left(\frac{n-\frac{2}{3\gamma}}{n-1}\right)^{n-1}}\left(\frac{t}{t_s}\right)^{\frac{2}{3\gamma}} \\
&+ \frac{a_s}{\frac{3\gamma}{2}\left(\frac{n-1}{n-\frac{2}{3\gamma}}\right)^{n-1}-1}\left(1-\frac{1-\frac{2}{3\gamma}}{n-\frac{2}{3\gamma}}\frac{t}{t_s}\right)^{n}, \qquad (17)
\end{aligned}
$$

with the admissible values of the parameters: $\gamma > 0$ and $n \neq 1$.

$w-$duality

We have a blow-up of the effective barotropic index, i.e.,

$$w(t_s) = \frac{c^2}{3}\left[2q(t_s) - 1\right] \to \infty \;, \tag{18}$$

accompanied by

$$p(t_s) \to 0; \quad \varrho(t_s) \to 0 \;. \tag{19}$$

There is an amazing duality between the Big-Bang and the w-singularity in the form

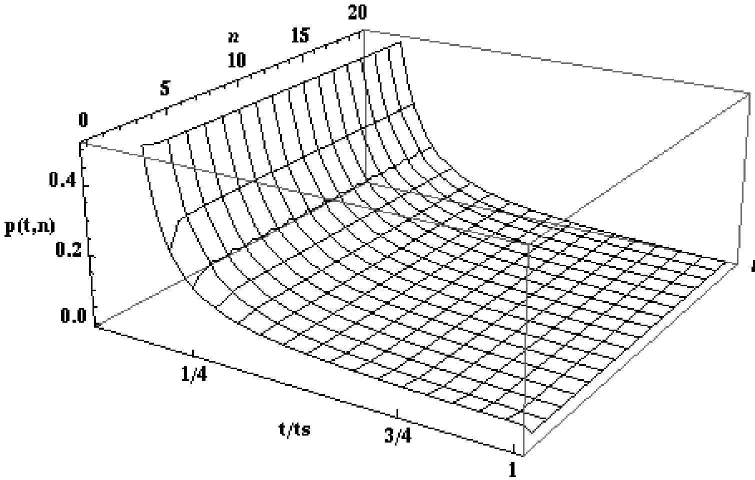
$$p_{BB} \leftrightarrow \frac{1}{p_w}, \quad \varrho_{BB} \leftrightarrow \frac{1}{\varrho_w}, \quad w_{BB} \leftrightarrow \frac{1}{w_w} \quad . \tag{20}$$

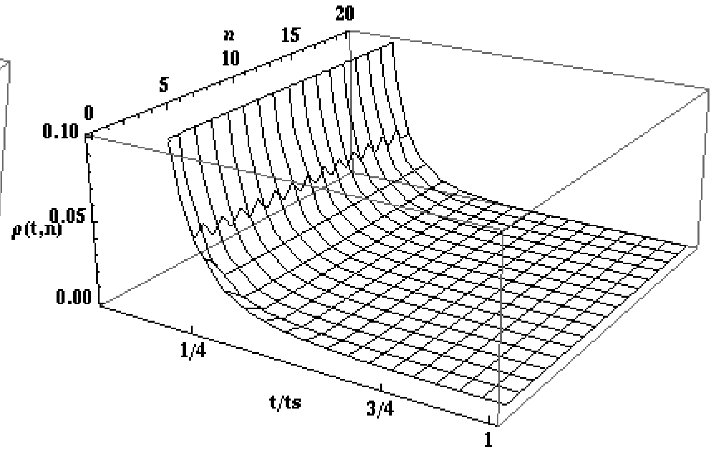
In other words:

$$p_{BB} \to \infty; \varrho_{BB} \to \infty; w_{BB} \to 0; a_{BB} \to 0$$

$$p_w \to 0; \varrho_w \to 0; w_w \to \infty; a_w \to a_s = \text{const.}$$

$w-$duality


p(t,n)
n
0
5
10
15
20
0.4
0.2
0.0
1/4
t/ts
3/4
1


ρ(t,n)
n
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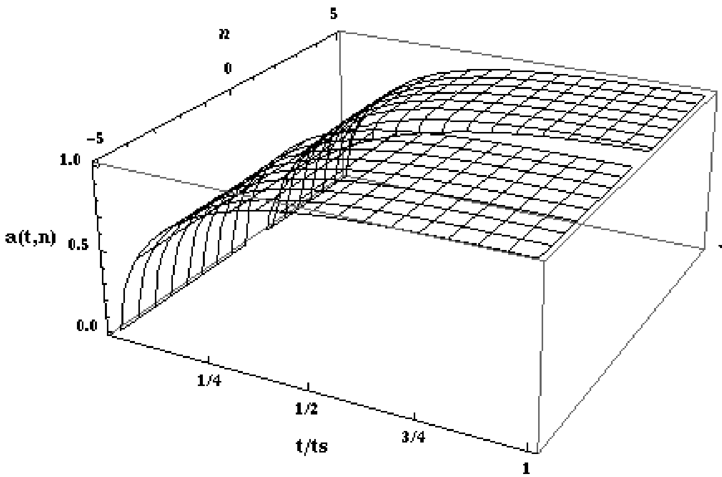
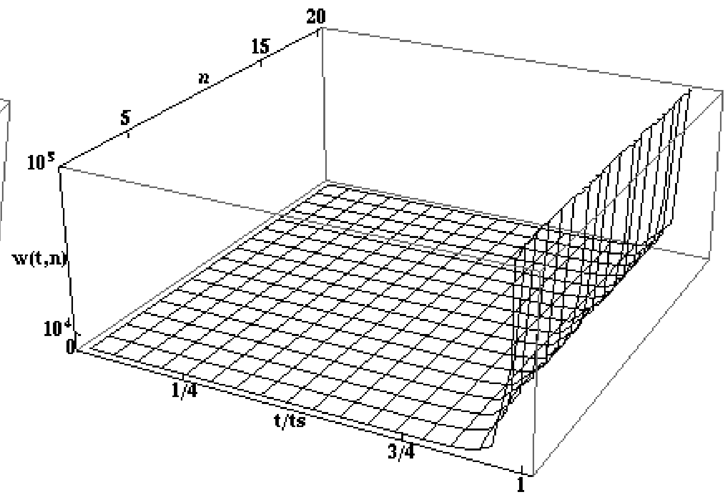
20
0.10
0.05
0.00
1/4
t/ts
3/4
1


a(t,n)
n
−5
0
5
1.0
0.5
0.0
1/4
1/2
3/4
1
t/ts


w(t,n)
n
5
15
20
10^5
10^4
0
1/4
t/ts
3/4
1

Classification of exotic singularities (Nojiri et al. 2005, MPD & Denkiewicz 2010).

- Type 0 - Big-Bang $a \to 0$, $p \to \infty$, $\varrho \to \infty$
- Type I - Big-Rip $a \to \infty$, $p \to \infty$, $\varrho \to \infty$ (Caldwell 2002)
- Type II - Sudden Future (includes Big Boost and Big-Brake) $a =$ const., $\varrho =$ const., $p \to \infty$ (Barrow 2004)
- Type IIg - Generalized Sudden Future a= const., $\varrho =$ const., $p =$const., $\dddot{a} \to \infty$ etc., $w < \infty$ (Barrow 2004)
- Type III - Finite Scale Factor (also Big-Freeze) $a = a_s$ = const., $\varrho \to \infty$, $p \to \infty$ (NOT 2005, Denkiewicz 2011)
- Type IV - Big Separation: a= const., $p = \varrho = 0$, $w \to \infty$, $\dddot{a} \to \infty$ etc. (NOT 2005) (and generalizations $p = \varrho =$const. Yurov 2010)
- Type V - w-singularity a= const., $p = \varrho = 0$, $w \to \infty$ (MPD, Denkiewicz 2009) (and generalizations $p =$const. Yurov 2010)
- Little-Rip, Pseudo-Rip (Frampton et al. 2011, 2012)

Are these really singularities - strength?

As an example let us take an SFS which is determined by a blow-up of the Riemann tensor and its derivatives.

Geodesics do not feel SFSs at all, since geodesic equations are not singular for $a_s = a(t_s) =$ const. (Fernandez-Jambrina, Lazkoz PRD 74, 064030 (2006))

$$\left(\frac{dt}{d\tau}\right)^2 = A + \frac{P^2 + KL^2}{a^2(t)} , \tag{21}$$

$$\frac{dr}{d\tau} = \frac{P_1 cos\phi + P_2 \sin\phi}{a^2(t)} \sqrt{1 - Kr^2} , \tag{22}$$

$$\frac{d\phi}{d\tau} = \frac{L}{a^2(t) r^2} . \tag{23}$$

Geodesic deviation equation

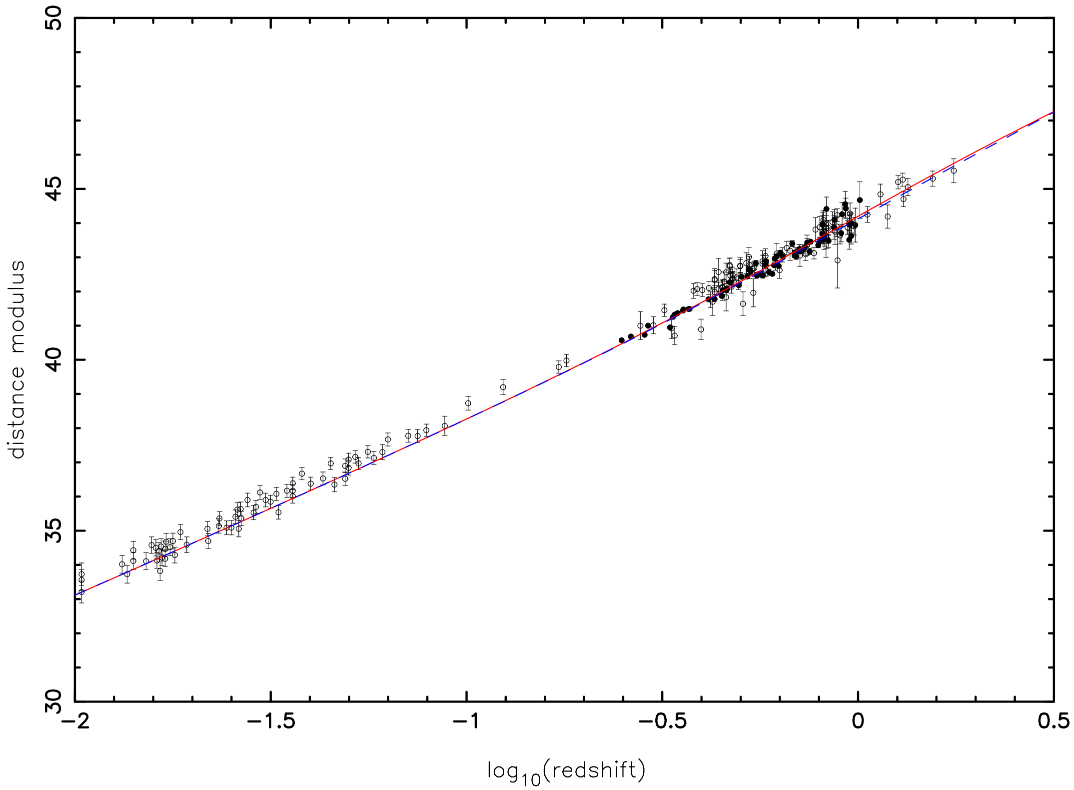
$$\frac{D^2 n^\alpha}{d\lambda^2} + R^\alpha_{\beta\gamma\delta} u^\beta n^\gamma u^\delta = 0 , \tag{24}$$

feels SFS since at $t = t_s$ we have the Riemann tensor $R^\alpha_{\beta\gamma\delta} \to \infty$.

Classification of exotic singularities - strength.

- Tipler's (Phys. Lett. A64, 8 (1977)) definition (of a weak singularity):
  $\int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' R_{ab} u^a u^b$
  does not diverge on the approach to a singularity at $\tau = \tau_s$
- Królak's (CQG 3, 267 (1988)) definition (of a weak singularity):
  $\int_0^{\tau} d\tau' R_{ab} u^a u^b$
  does not diverge on the approach to a singularity at $\tau = \tau_s$
- Type 0 (BB, BC): T, K - strong
- Type I (BR): T, K - strong
- Type II (SFS): T, K - weak
- Type IIg (GSFS): T, K - weak
- Type III (FSF): T - weak, K - strong
- Type IV (BS): T, K - weak
- Type V (w-sing.): T, K - weak (Fernandez-Jambrina (PRD, 2010)

SFS dark energy mimics Λ-term (supernovae only)



Distance modulus $\mu_L = m - M$ for the CC model ($H_0 = 72\text{kms}^{-1}\text{Mpc}^{-1}$, $\Omega_{m0} = 0.26$, $\Omega_{\Lambda 0} = 0.74$) (dashed curve) and SFS model ($m = 2/3 = 0.6666, n = 1.9999, \delta = -0.471, y_0 = 0.99936$) (solid curve). Open circles are for the ‘Gold’ data and filled circles are for SNLS data.

CMB shift parameter.

It is possible to fit other tests but at the expense of relaxing the range of the parameter m which refers to Big-Bang limit ($m = 2/3$ is dust).

Shift parameter is:

$$\mathcal{R} = \frac{l_1^{\prime TT}}{l_1^{TT}} \tag{25}$$

where

l_1^{TT} – the temperature perturbation CMB spectrum multipole of the first acoustic peak in SFS model

$l_1^{\prime TT}$ – the multipole of a reference flat standard Cold Dark Matter model.

One usually uses a rescaled shift parameter:

$$\mathcal{R} = \frac{H_0 a_0}{c}\sqrt{\Omega_{m0}} r_{dec} = \sqrt{\Omega_{m0}} a'(y) \int_{y_{dec}}^{y_0} \frac{dy}{a(y)} = \sqrt{\Omega_{m0}} \int_0^{z_{dec}} \frac{dz}{E(z)}, \tag{26}$$

and WMAP data gives $\mathcal{R} = 1.70 \pm 0.03$ (Wang et al. 2006).

Baryon acoustic oscillations.

The Alcock-Paczyński effect says that one is able to calculate the distortion of a spherical object in the sky without knowing its true size.
This can be done by measuring its transverse extend (using the angular diameter distance $d_A = l/\Delta\theta$, where l is the linear size of an object) and line-of-sight extend (using the redshift distance $\Delta x = c\Delta t/a(t) = ct_s\Delta y/a(y)$) (see e.g. Nesseris et al. 2006). As a result one defines the volume distance as

$$D_V^3 = d_A^2 \Delta x \quad , \tag{27}$$

so that one has

$$D_V = \left[\left(\int_{y_1}^{y_0} \frac{ct_s dy}{a(y)} \right)^2 \left(\frac{ct_s \Delta y}{a(y)} \right) \right]^{\frac{1}{3}} = \left[\left(\frac{c}{a_0 H_0} \int_0^z \frac{dz}{E(z)} \right)^2 \left(\frac{c}{a_0 H_0} \frac{\Delta z}{E(z)} \right) \right]^{\frac{1}{3}} . \tag{28}$$

Eisenstein et al. (2005) gave $D_V(\Delta z = z_{BAO} = 0.35) = 1370 \pm 64$ Mpc (an acoustic peak for 46748 luminous red galaxies (LRG) selected from the SDSS (Sloan Digital Sky Survey).

Baryon acoustic oscillations - dimensionless parameter A.

For our sudden future singularity model (14) it is more convenient to use a dimensionless quantity $\mathcal{A}$ which is obtained multiplying D_V by $\sqrt{\Omega_{m0}}/(ct_s z_{BAO})$ or by $\sqrt{\Omega_{m0}}(a_0 H_0)/(c z_{BAO})$ to get

$$\mathcal{A} = \sqrt{\Omega_{m0}} a'(y_0) \left[\frac{a(y_{BAO})}{a'(y_{BAO}) a(y_0)} \right]^{\frac{1}{3}} \left[\frac{1}{z_{BAO}} \int_{y_{BAO}}^{y_0} \frac{dy}{a(y)} \right]^{\frac{2}{3}} \tag{29}$$

or

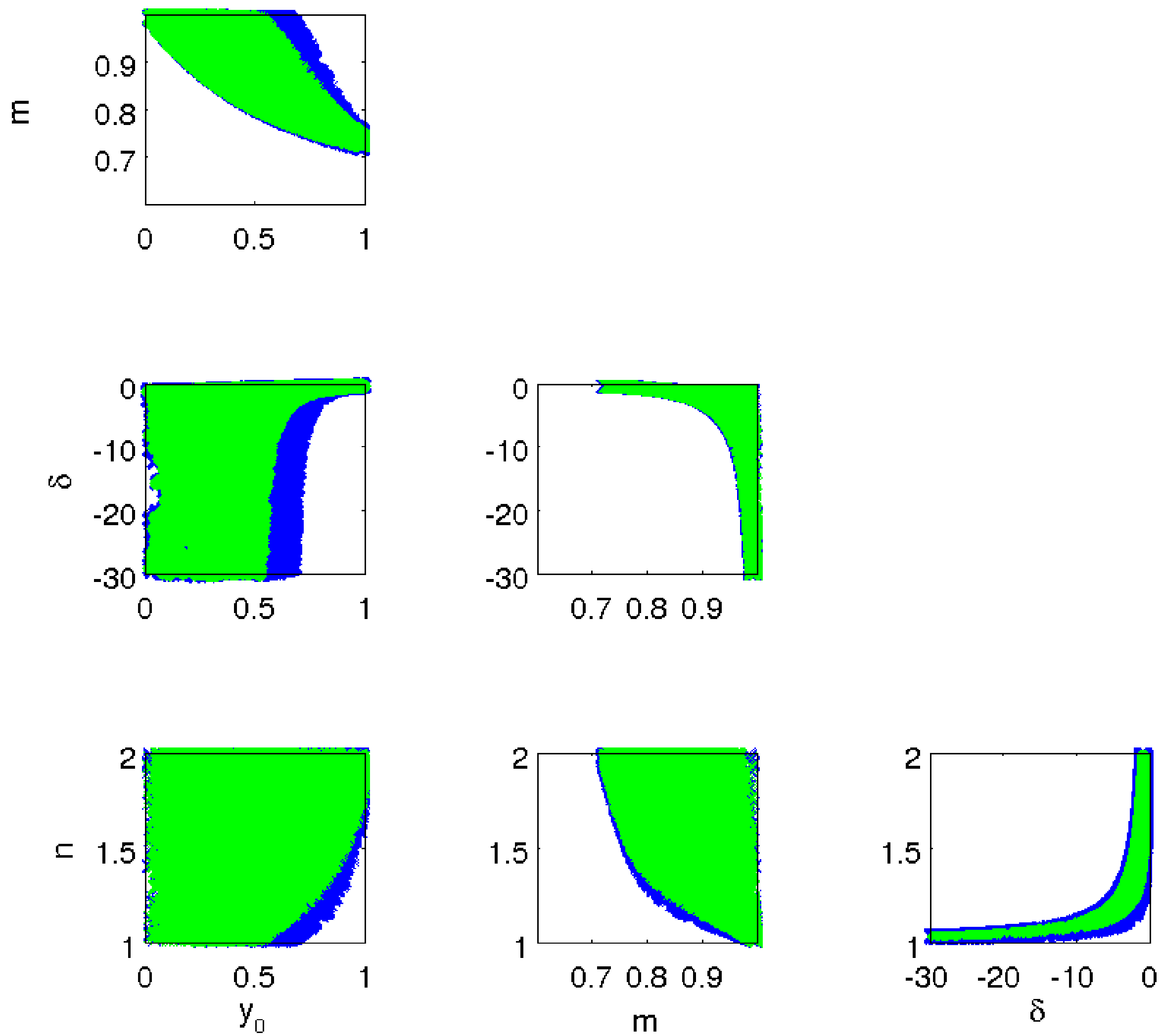
$$\mathcal{A} = \sqrt{\Omega_{m0}} E(z_{BAO})^{-1/3} \left[\frac{1}{z_{BAO}} \int_0^{z_1} \frac{dz}{E(z)} \right]^{2/3} \tag{30}$$

It should have the value (Eisenstein et al. 2005)

$$\mathcal{A} = 0.469 \left(\frac{n}{0.98} \right)^{-0.35} \pm 0.017 \ , \tag{31}$$

where n is the spectral index (now taken about ~ 0.96).

Combined bound: supernovae, CMB shift parameter and BAO - fits if $m \approx 0.72$, $w = -0.82$. (Denkiewicz et al. 2012)



4. The universe through a singularity - averaging surprises.

A.K. Raychaudhuri (PRL 80, 654 (1998)) proposed that one may average physical and kinematical scalars over the whole open spacetime provided they vanish rapidly at spatial and temporal infinity as follows

$$< \chi >= \lim_{x^a \to \infty} \frac{\int\int\int\int_{-x^a}^{x^a} \chi \sqrt{-g} d^4x}{\int\int\int\int_{-x^a}^{x^a} \sqrt{-g} d^4x} \quad (32)$$

By an open model it is meant that the ratio of the 3-volume hypersurfaces to a 4-volume of spacetime vanishes, i.e.,

$$\frac{\int\int\int \sqrt{|^3 g|} d^3x}{\int\int\int\int \sqrt{-g} d^4x} = 0. \quad (33)$$

His idea was to tight the vanishing of the average $< \chi >$ with the singularity avoidance in cosmology.

Spacetime averaging - density and pressure.

For the pressure, the energy density, and the average acceleration we have

$$< p >= - \lim_{\substack{t_0 \to 0 \\ t_1 \to \infty}} \frac{\int_{t_0}^{t_1} a^3 \left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) dt}{\int_{t_0}^{t_1} a^3 dt} \tag{34}$$

and

$$< \varrho >= \lim_{\substack{t_0 \to 0 \\ t_1 \to \infty}} \frac{3 \int_{t_0}^{t_1} a^3 \left(\frac{\dot{a}^2}{a^2} \right) dt}{\int_{t_0}^{t_1} a^3 dt}. \tag{35}$$

$$< \dot{\theta} >= \lim_{\substack{t_0 \to 0 \\ t_1 \to \infty}} \frac{3 \int_{t_0}^{t_1} a^3 \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) dt}{\int_{t_0}^{t_1} a^3 dt}. \tag{36}$$

SFS universe through an exotic singularity.

One is able to construct a hybrid model which allows Big-Bang, Sudden Future Singularity and finally Big-Crunch given by:

$$a_L(t) = a_s \left[\delta + \left(1 + \frac{t}{t_B} \right)^m (1-\delta) - \delta \left(-\frac{t}{t_B} \right)^n \right] \quad (37)$$

with $t_B < 0$ - the Big-Bang time, and $t = 0$ and SFS time;

$$a_R(t) = a_s \left[\delta + \left(1 - \frac{t}{t_C} \right)^m (1-\delta) - \delta \left(\frac{t}{t_C} \right)^n \right] \quad (38)$$

with $t_C > 0$ - the Big-Crunch time. In the high pressure regime $t \to 0$ these are approximated by

$$a_L \approx a_s \left[1 + \frac{m}{t_B} (1-\delta)\, t \right], \quad (39)$$

$$a_R \approx a_s \left[1 - \frac{m}{t_C} (1-\delta)\, t \right]. \quad (40)$$

Spacetime averaging - standard and phantom models

$$< p >_{stand} = \lim_{\substack{t_0 \to 0 \\ t_1 \to \infty}} -\frac{4}{\gamma}\left(\frac{1}{\gamma} - 1\right)\frac{\int_{t_0}^{t_1} t^{2\left(\frac{1}{\gamma}-1\right)} dt}{\int_{t_0}^{t_1} t^{\frac{2}{\gamma}} dt} \to 0,$$

$$< \varrho >_{stand} = \lim_{\substack{t_0 \to 0 \\ t_1 \to \infty}} -\frac{4}{3\gamma^2}\frac{\int_{t_0}^{t_1} t^{2\left(\frac{1}{\gamma}-1\right)} dt}{\int_{t_0}^{t_1} t^{\frac{2}{\gamma}} dt} \to 0$$

$$< p >_{ph} = \lim_{\substack{t_0 \to 0 \\ t_1 \to \infty}} -\frac{4}{\mid \gamma \mid}\left(\frac{1}{\mid \gamma \mid} + 1\right)\frac{\int_{t_0}^{t_1} t^{-2\left(\frac{1}{|\gamma|}+1\right)} dt}{\int_{t_0}^{t_1} t^{-\frac{2}{|\gamma|}} dt} \to \infty,$$

$$< \varrho >_{ph} = \lim_{\substack{t_0 \to 0 \\ t_1 \to \infty}} -\frac{4}{3\gamma^2}\frac{\int_{t_0}^{t_1} t^{-2\left(\frac{1}{|\gamma|}+1\right)} dt}{\int_{t_0}^{t_1} t^{-\frac{2}{|\gamma|}} dt} \to \infty$$

Spacetime averaging - SFS and FSF models

$$\dot{a}_L(t) = a_s \left[\frac{m}{t_B} \left(1 + \frac{t}{t_B} \right)^{m-1} (1-\delta) + \delta \frac{n}{t_B^n} (-t)^{n-1} \right] \tag{41}$$

$$\dot{a}_R(t) = a_s \left[-\frac{m}{t_C} \left(1 - \frac{t}{t_C} \right)^{m-1} (1-\delta) + \delta \frac{n}{t_C^n} (t)^{n-1} \right] \tag{42}$$

$$\frac{\ddot{a}_L}{a_s} = \frac{m(m-1)(1-\delta)}{t_B^2} \left(1 + \frac{t}{t_B} \right)^{m-2} - \frac{\delta n(n-1)}{t_B^n} (-t)^{n-2} \tag{43}$$

$$\frac{\ddot{a}_R}{a_s} = \frac{m(1-m)(1-\delta)}{t_C^2} \left(1 - \frac{t}{t_C} \right)^{m-2} + \frac{\delta n(n-1)}{t_C^n} t^{n-2} \tag{44}$$

Only the last terms blow up to give infinite pressure for $1 < n < 2$ at $t = 0$ so that we neglect other terms in a, $\dot{a}$ and $\ddot{a}$.

Spacetime averaging - SFS and FSF models

$$< \dot{\theta} >_{SFS,L} = \lim_{\substack{t_0 \to -t_B \\ t_1 \to 0}} -3n \frac{\int_{t_0}^{t_1} (-t)^{3n-2} dt}{\int_{t_0}^{t_1} (-t)^{3n} dt} \tag{45}$$

$$= \lim_{\substack{t_0 \to -t_B \\ t_1 \to 0}} -3n \frac{3n+1}{3n-1} \frac{(-t_1)^{3n-1} - (-t_0)^{3n-1}}{(-t_1)^{3n+1} - (-t_0)^{3n+1}} \to \frac{1}{t_B^2}$$

$$< \dot{\theta} >_{SFS,R} = \lim_{\substack{t_0 \to 0 \\ t_1 \to t_C}} 3n \frac{\int_{t_0}^{t_1} t^{3n-2} dt}{\int_{t_0}^{t_1} t^{3n} dt} \tag{46}$$

$$= \lim_{\substack{t_0 \to 0 \\ t_1 \to t_C}} 3n \frac{3n+1}{3n-1} \frac{t_1^{3n-1} - t_0^{3n-1}}{t_1^{3n+1} - t_0^{3n+1}} \to \frac{1}{t_C^2}$$

These averages are finite for SFS, but they may blow up for FSF if $0 < n < 1/3$!

Subtle differences between singularities.

- BB, BC singularities - all the energy conditions fulfilled, averages vanish (despite original claim of Raychaudhuri)
- BR singularity - no EC fulfilled, averages blow up
- SFS - only dominant energy violated, averages finite
- It seems that BR is stronger singularity that BB, BC on the ground of averaging.
- SFS is weaker, but FSF does not seem so.

This seems to be a new kind of a measure for the strength of singularities.

4. Varying constants cosmologies and their advantages.

Problem: It has been shown that quantum effects (e.g. Houndjo et al. arXiv:1203.6084) may change the strength of exotic singularities. We ask if this is also the case once one assumes variability of physical constants?

Pretty long story of varying constants theories:

Hermann Weyl (1919): electron radius/its gravitational radius $\sim 10^{40}$

Arthur Eddington (1935) discussed:

1) proton-to-electron mass $1/\beta = m_p/m_e \sim 1840$

2) an inverse of fine structure constant $1/\alpha = (hc)/(2\pi e^2) \sim 137$

3) electromagnetic to gravitational force between a proton and an electron $e^2/(4\pi\epsilon_0 G m_e m_p) \sim 10^{40}$

4) introduced "Eddington number" $N_{edd} \sim 10^{80}$

P.A.M. Dirac (1937) interesting remarks about the relations between atomic and cosmological quantities: If $G \propto H(t) = (da/dt)/a$, then $a(t) \propto t^{1/3}$ and $G(t) \propto 1/t$ - fundamental constants must evolve in time.

contd. varying constant cosmologies and their advantages.

First fully quantitative framework: Brans-Dicke scalar-tensor gravity (1961)

The gravitational constant G is associated with an average gravitational potential (scalar field) ϕ surrounding a given particle:

$< \phi >= GM/(c/H_0) \propto 1/G = 1.35 \times 10^{28} g/cm$. The scalar field gives the strength of gravity

$$G = \frac{1}{16\pi\Phi} \tag{47}$$

With the action

$$S = \int d^4x\sqrt{-g}\left(\Phi R - \frac{\omega}{\Phi}\partial_\mu\Phi\partial^\mu\Phi + \Lambda + L_m\right) \tag{48}$$

it relates to low-energy-effective superstring theory for $\omega = -1$

String coupling constant (running) $g_s = \exp{(\phi/2)}$ changes in time with ϕ - the dilaton and $\Phi = \exp{(-\phi)}$.

contd. varying constants cosmologies and their advantages.

Varying speed of light theories (VSL): Albrecht & Magueijo model (AM model) (1999)(Barrow 1999; Magueijo 2003):

$$c^4 = \psi(x^\mu) \tag{49}$$

and so the action is

$$S = \int d^4x \sqrt{-g} \left[\frac{\psi(R + 2\Lambda)}{16\pi G} + L_m + L_\psi \right] \tag{50}$$

AM model breaks Lorentz invariance (relativity principle and light principle) - preferred frame (cosmological or CMB) in which the field is minimally coupled to gravity.

Solves basic problems of standard cosmology: horizon problem and flatness problem.

Ansatz: Friedmann with $\rho = \rho_0 a^{-3\gamma}$, $c(t) = c_0 a^n$ - solution if $n \leq (1/2)(2 - 3\gamma)$.

contd. varying constants cosmologies and their advantages.

Magueijo covariant (conformally) and locally invariant model (2000, 2001):

$$\psi = \ln\left(\frac{c}{c_0}\right) \quad \text{or} \quad c = c_0 e^{\psi} \, , \tag{51}$$

with the action

$$S = \int d^4x\sqrt{-g}\left[\frac{c_0^4 e^{\alpha\psi}(R + 2\Lambda + L_\psi)}{16\pi G} + e^{\beta\psi}L_m\right] \, , \tag{52}$$

with

$$L_\psi = \kappa(\psi)\nabla_\mu\psi\nabla^\mu\psi \, . \tag{53}$$

Further assumption: $\alpha - \beta = 4$.

Interesting subcases:

$\alpha = 4; \beta = 0$ - Brans-Dicke with $\phi_{BD} = e^{4\psi}/G$ and $\kappa(\psi) = 16\omega_{BD}(\phi_{BD})$.

$\alpha = 0; \beta = -4$ - minimal VSL theory.

contd. varying constants cosmologies and their advantages.

Varying fine structure constant α (or charge $e = e_0\epsilon(x^\mu)$ theories (Webb et al. 1999, Sandvik 2002)

$$S = \int d^4x\sqrt{-g}\left(\psi R - \frac{\omega}{2}\partial_\mu\psi\partial^\mu\psi - \frac{1}{4}f_{\mu\nu}f^{\mu\nu}e^{-2\psi} + L_m\right) \tag{54}$$

with $\psi = \ln\epsilon$ and $f_{\mu\nu} = \epsilon F_{\mu\nu}$.
Assume linear expansion $e^\psi = 1 - 8\pi G\zeta(\psi - \psi_0) = 1 - \Delta\alpha/\alpha$ with the constraint on the local equivalence principle violence $\mid \zeta \mid \leq 10^{-3}$. The relation to DE is:

$$\gamma = \frac{(8\pi G\frac{d\psi}{d\ln a})^2}{\Omega_\psi} \tag{55}$$

This can be tested while mimicking the dark energy by spectrograph CODEX (COsmic Dynamics EXplorer) a device attached to planned E-ELT (European Extremely Large Telescope) measuring the so-called redshift drift (or Sandage-Loeb effect) for $2 < z < 5$ (Vielzeuf and Martins 2012).

Observational constraints:

- $|(dG/dt)/G| < 9 \cdot 10^{-13}/year$ - from primordial nucleosynthesis (Accetta et al. 1990);
- $|(dG/dt)/G| < 1.6 \cdot 10^{-12}/year$ - from helioseismology (Guenther et al. 1998);
- $|(dG/dt)/G| < (4 \pm 9) \cdot 10^{-13}/year$ - from lunar laser ranging (LLR) (Williams et al. 1996);
- $\Delta\alpha/\alpha = (3.85 \pm 5.65) \cdot 10^{-8}$ - from Oklo phenomenon (Shlyakhter 1976, Petrov et al. 2006);
- $\Delta\alpha/\alpha = (-8 \pm 16) \cdot 10^{-7}$ - from meteorite dating (long-lived beta decays) (Olive et al. 2003);
- $\Delta\alpha/\alpha = (-0.5 + / - 1.3) \cdot 10^{-5}$ - from quasar absorption spectra with redshifts $2.33 < z < 3.08$ (Murphy et al. 2001);

Where the fine structure constant involves variability of $c(t)$: $\alpha = e^2/4\pi\hbar c$.

5. Varying constants versus cosmic singularities.

We consider the Friedmann universes in varying speed of light (VSL) theories and varying gravitational constant G theories as follows (ϱ - mass density; $\varepsilon = \varrho c^2(t)$ - energy density in $Jm^{-3} = Nm^{-2} = kgm^{-1}s^{-2}$)

$$\varrho(t) = \frac{3}{8\pi G(t)}\left(\frac{\dot{a}^2}{a^2} + \frac{kc^2(t)}{a^2}\right) , \tag{56}$$

$$p(t) = -\frac{c^2(t)}{8\pi G(t)}\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{kc^2(t)}{a^2}\right) , \tag{57}$$

and the energy-momentum conservation law is

$$\dot{\varrho}(t) + 3\frac{\dot{a}}{a}\left(\varrho(t) + \frac{p(t)}{c^2(t)}\right) = -\varrho(t)\frac{\dot{G}(t)}{G(t)} + 3\frac{kc(t)\dot{c}(t)}{4\pi G a^2} . \tag{58}$$

New form of the scale factor.

We propose a new form of the scale factor, which admits big-bang, big-rip, sudden future, finite scale factor and w-singularities and reads as

$$a(t) = a_s \left(\frac{t}{t_s}\right)^m \exp\left(1 - \frac{t}{t_s}\right)^n , \qquad (59)$$

with the constants t_s, a_s, m, n. For $k = 0$ we have

$$\varrho(t) = \frac{3}{8\pi G(t)} \left[\frac{m}{t} - \frac{n}{t_s}\left(1 - \frac{t}{t_s}\right)^{n-1}\right]^2 , \qquad (60)$$

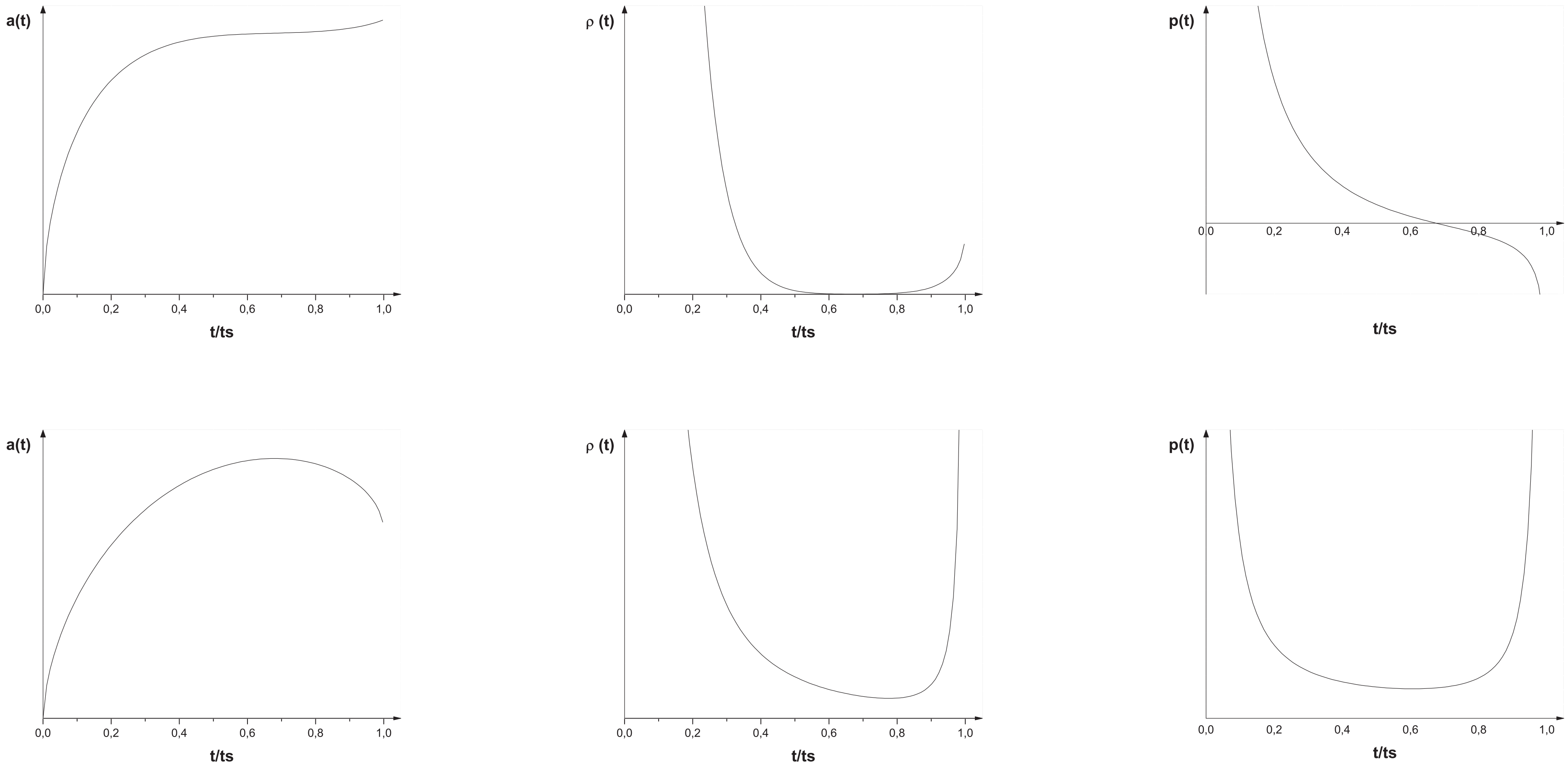
$$p(t) = -\frac{c^2(t)}{8\pi G(t)} \left[\frac{m(3m-2)}{t^2} - 6\frac{mn}{tt_s}\left(1 - \frac{t}{t_s}\right)^{n-1} \right. \qquad (61)$$

$$\left. + \; 3\frac{n^2}{t_s^2}\left(1 - \frac{t}{t_s}\right)^{2(n-1)} + 2\frac{n(n-1)}{t_s^2}\left(1 - \frac{t}{t_s}\right)^{n-2}\right] .$$

Contd. - new form of the scale factor.

For $0 < m < 2/3$ we have a big-bang singularity - $a \to 0$, $\varrho \to \infty$, $p \to \infty$ at $t \to 0$;
For $m < 0$ we have a big-rip singularity - $a \to \infty$, $\varrho \to \infty$, $p \to \infty$ at $t = 0$;
For $1 < n < 2$ we have a sudden future singularity (SFS) which appears at $t = t_s$ ($a = a_s$, $\varrho =$ const., $p \to \infty$);
For $0 < n < 1$ we have a stronger finite scale factor singularity (FSF) at $t = t_s$ ($a = a_s$, $\varrho \to \infty$, $p \to \infty$).
The plots of the scale factor $a(t)$, the energy density $\varrho(t)$, and the pressure $p(t)$ are given in Fig. (next page) for the two specific models. The upper plots are for the parameters $m = 0.6$, $n = 1.5$ and describe the sudden future singularity (SFS) while lower plots are for the parameters $m = 0.6$ and $n = 0.5$ and describe the finite scale factor singularity (FSF).
In fact, for $1 < n < 2$ only the last term in the pressure of the type $(1 - t/t_s)^{n-2}$ blows-up, while for $0 < n < 1$ two more terms $(1 - t/t_s)^{n-1}$ and $(1 - t/t_s)^{2(n-1)}$ do.

New form of the scale factor - plots



Regularizing singularities by varying constants

New idea: to change or even regularize various cosmological singularities by the variation of physical constants such as G, c, α etc.

One bears in mind the scale factor (59), the energy density (60) and pressure (61)

Regularizing a Big-Bang singularity by varying G:

If

$$G(t) \propto \frac{1}{t^2} \tag{62}$$

which is a faster decrease than in Dirac's LNH $G \propto 1/t$, but influences less the temperature of the Earth constraint (Teller 1948). Both divergence in ϱ and p are removed, though at the expense of having the "singularity" of strong gravitational coupling $G \to \infty$ at $t \to 0$. Besides, in the Dirac's case, only the ϱ singularity can be removed.

contd. - regularizing singularities by varying constants: SFS

Regularizing an SFS singularity by varying c:

If

$$c(t) = c_0 \left(1 - \frac{t}{t_s}\right)^{\frac{p}{2}} , \tag{63}$$

then

$$p(t) = -\frac{c_0^2}{8\pi G}\left[\frac{m(3m-2)}{t^2}\left(1-\frac{t}{t_s}\right)^{p} - 6\frac{mn}{tt_s}\left(1-\frac{t}{t_s}\right)^{p+n-1} + 3\frac{n^2}{t_s^2}\left(1-\frac{t}{t_s}\right)^{p+2n-2} + 2\frac{n(n-1)}{t_s^2}\left(1-\frac{t}{t_s}\right)^{p+n-2}\right].$$

and the singularity of pressure is regularized provided $p > 2-n, (1 < n < 2)$. Physical consequence: light eventually stops at the singularity. Same happens in loop quantum cosmology (LQC) where it is called the anti-newtonian limit $c = \sqrt{1-\varrho/\varrho_c} \to 0$ for $\varrho \to \varrho_c$ with ϱ_c being the critical density (Cailettau et al. 2012). The low-energy limit $\varrho \ll \varrho_0$ gives the standard limit $c \to 1$.

contd. - regularizing singularities by varying constants: w-sing.

It also appears naturally in Magueijo model (2001) in which black holes are not reachable since the light stops at the horizon (despite they possess Schwarzschild singularity). Besides, both options $c = 0$ and $c = \infty$ are possible in this model.

contd. - regularizing singularities by varying constants: w-sing.

In the limit $m \to 0$ we have an exotic singularity scale factor given by $a(t) = a_s \exp\left(1 - t/t_s\right)$ and so from (60) and (61) we have

$$\varrho_{ex}(t) = \frac{3}{8\pi G(t)} \frac{n^2}{t_s^2} \left(1 - \frac{t}{t_s}\right)^{2(n-1)} , \quad (64)$$

$$p_{ex}(t) = -\frac{c^2(t)}{8\pi G(t)} \left[3\frac{n^2}{t_s^2} \left(1 - \frac{t}{t_s}\right)^{2(n-1)} + 2\frac{n(n-1)}{t_s^2} \left(1 - \frac{t}{t_s}\right)^{n-2} \right] \quad (65)$$

so that

$$w_{ex}(t) = \frac{p_{ex}(t)}{\varepsilon_{ex}(t)} = - \left[1 + \frac{2}{3} \frac{n-1}{n} \frac{1}{\left(1 - \frac{t}{t_s}\right)^n} \right] = - \left[\frac{1}{3} - \frac{2}{3} q_{ex}(t) \right] , \quad (66)$$

which is a w-singularity for $n > 2$ ($p = \varrho = 0$, $w_{ex} \to \infty$). Its regularization by varying $c(t)$ is impossible since there is no c-dependence here.

contd. - regularizing singularities by varying constants: SFS

Regularizing an SFS singularity by varying G:

If we assume that

$$G(t) = G_0 \left(1 - \frac{t}{t_s}\right)^{-r} , \tag{67}$$

($r =$ const., $G_0 =$ const.) which changes (60) and (61) to

$$\begin{aligned} \varrho(t) &= \frac{3}{8\pi G_0}\left[\frac{m^2}{t^2}\left(1-\frac{t}{t_s}\right)^{r} - \frac{2mn}{tt_s}\left(1-\frac{t}{t_s}\right)^{r+n-1}\right. \\ &+ \left.\frac{n^2}{t_s^2}\left(1-\frac{t}{t_s}\right)^{r+2n-2}\right] , \end{aligned} \tag{68}$$

$$\begin{aligned} p(t) &= -\frac{c^2}{8\pi G_0}\left[\frac{m(3m-2)}{t^2}\left(1-\frac{t}{t_s}\right)^{r} - 6\frac{mn}{tt_s}\left(1-\frac{t}{t_s}\right)^{r+n-1} + 3\frac{n^2}{t_s^2}\left(1-\frac{t}{t_s}\right)^{r+2n-2}\right. \\ &+ \left. 2\frac{n(n-1)}{t_s^2}\left(1-\frac{t}{t_s}\right)^{r+n-2}\right] . \end{aligned} \tag{69}$$

contd. - regularizing singularities by varying constants: SFS

From (68) and (69) it follows that an SFS singularity $(1 < n < 2)$ is regularized by varying gravitational constant when

$$r > 2 - n \ , \tag{70}$$

and an FSF singularity $(0 < 1 < n)$ is regularized when

$$r > 1 - n \ . \tag{71}$$

On the other hand, assuming that we have an SFS singularity and that

$$-1 < r < 0 \, , \tag{72}$$

we get that varying G may change an SFS singularity onto a stronger FSF singularity when

$$0 < r + n < 1 \, . \tag{73}$$

Regularizing singularities: (anti-)Chaplygin gas

The equation of state of the (anti-)Chaplygin gas reads as

$$p(t) = \pm \frac{A}{\varepsilon(t)} = \pm \frac{A}{\varrho(t)c^2(t)} \qquad (A > 0) \ , \tag{74}$$

where the "-" sign is for Chaplygin gas while the "+" sign is for anti-Chaplygin gas case and the unit of A is the energy density(=pressure) square $J^2 m^{-6}$. Inserting (74) into (58) gives

$$\dot{\varrho}(t) + 3\frac{\dot{a}}{a}\left(\frac{\varrho^2 c^4(t) \mp A}{\varrho(t)c^4(t)}\right) = -\varrho(t)\frac{\dot{G}(t)}{G(t)} + 3\frac{kc(t)\dot{c}(t)}{4\pi G(t)a^2} \ . \tag{75}$$

We assume both varying $G = G(t)$ and $c = c(t)$ though with zero curvature $(k = 0)$ as follows

$$\varrho(t)c^2(t) = B = \text{const.} \ , \tag{76}$$

contd. - regularizing singularities: (anti-)Chaplygin gas

The solution of (75) reads as

$$\varrho(t)a^{3\gamma}(t)G(t) = E = \text{const.} \; , \tag{77}$$

where we have defined

$$\gamma \equiv \frac{B^2 \mp A}{B^2} \tag{78}$$

Putting the standard big-bang scale factor $a(t) = (t/t_s)^{2/3\gamma}$, we finally have

$$\varrho(t) = \frac{Et_s^2}{t^2 G(t)} \; , \qquad p(t) = \mp \frac{A}{B} = \text{const.} \; , \tag{79}$$

which give $\varrho \to \infty$ and $p(0) = 0$ provided $G(0) = \text{const.} \neq 0$. The singularity at $t = 0$ in ϱ and p can be regularized by taking $G(t) \propto 1/t^2$ at the expense of having a constant pressure (cosmological term) instead of zero pressure.

Physical subtleties:

- In order to regularize an SFS or an FSF singularity by varying $c(t)$, the light should slow and eventually stop propagating at a singularity. Similar effects were found in loop quantum cosmology (LQC).
- To regularize an SFS, FSF by varying gravitational constant $G(t)$ - the strength of gravity has to become infinite at a singularity. On the one hand, it is quite reasonable because of the requirement to overcome an infinite (anti-)tidal forces at the singularity, but on the other hand, it makes another singularity - a singularity of strong coupling for a physical field such as $G \propto 1/\Phi$. Such problems were already dealt with in superstring and brane cosmology where both the curvature singularity and a strong coupling singularity appeared (choice of coupling, quantum corrections).

5. Conclusions

- Currently one is able to differentiate quite a number of cosmological singularities with completely different properties - despite many of them are geodesically complete, they still lead to a blow-up of various physical quantities (scale factor, energy density, pressure, physical fields).
- Some of these singularities may serve as dark energy, especially if they are quite close in the near future. For example, an SFS may even appear in 8.7 Myr with no contradiction with bare supernovae data. It can be fitted to a combined SnIa, CMB and BAO data, but at the expense of admitting an approach to a Big-Bang by a fluid which is not exactly dust (m=0.66), but has a slightly negative pressure (m = 0.73 and so w = -0.09).
- An interesting proposal is to investigate how the singularities are influenced by varying physical constants. In particular, we may look for the answer if it is possible to "regularize" (remove infinities) or change these singularities and what are the physical consequences of such an action, because what we face is usually the new "singularity" in a physical constant/field which acts to remove/change the type of singularity.