Standard and exotic singularities regularized by varying constants

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Problem:

over the last decade a lot of exotic (non-big-bang) types of singularities have been uncovered - how can they be influenced by the variability of the fundamental constants?

Content:

1. Introduction.

Standard Einstein-Friedmann equations are two equations for three unknown functions of time $a(t), p(t), \varrho(t)$

$$\varrho = \frac{3}{8\pi G}\left(\frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) , \quad (1)$$

$$p = -\frac{1}{8\pi G}\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) . \quad (2)$$

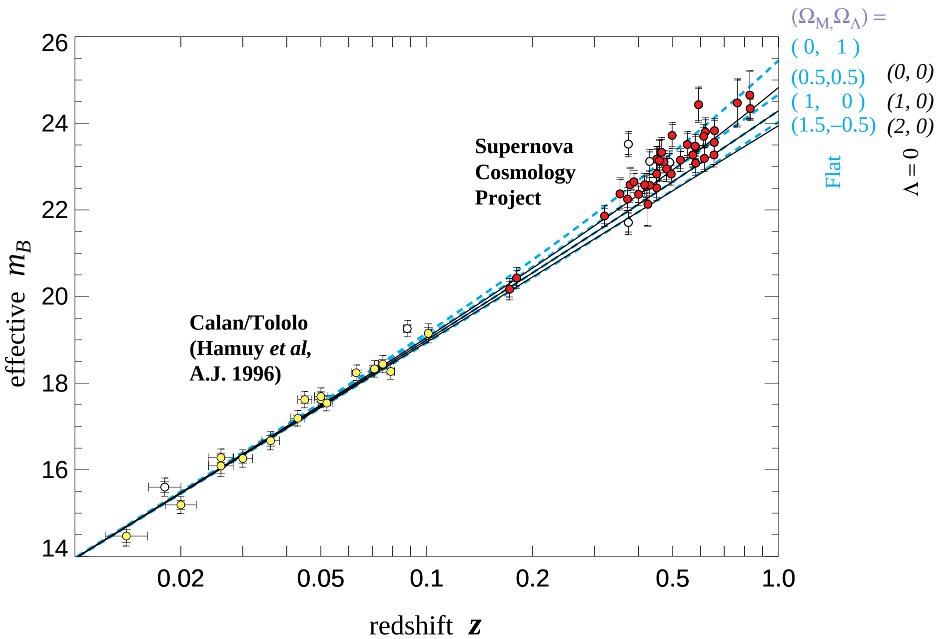
plus an equation of state, e.g., of a barotropic type ($w =$ const. ≥ -1):

$$p(t) = w\varrho(t) \qquad \to a(t) \propto t^{\frac{2}{3(w+1)}} . \quad (3)$$

Until very recently (including first supernovae results) most of cosmologists studied only simplest - say **"standard" solutions - each of them starts with Big-Bang singularity** in which $a \to 0$, $\varrho, p \to \infty$
– one of them (of $K = +1$) terminates at the second singularity (Big-Crunch) where $a \to 0$, $\varrho, p \to \infty$
– the other two ($K = 0, -1$) continue to an asymptotic emptiness $\varrho, p \to 0$ for $a \to \infty$.

BB and BC exhibit geodesic incompleteness and curvature blow-up.

However, first supernovae observations ...



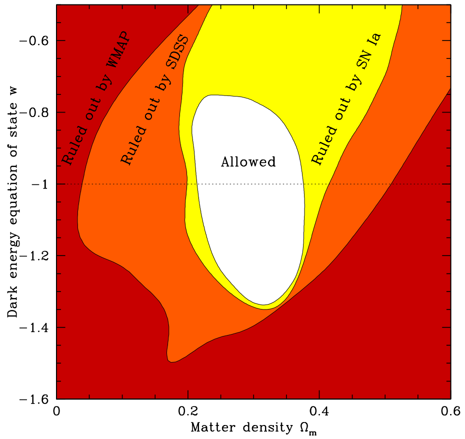
... gave evidence for the strong energy condition

$$\varrho + 3p \geq 0, \qquad \varrho + p \geq 0\ . \tag{4}$$

violation, but **the paradigm of the “standard” Big-Bang/Crunch singularities remained untouched.**

2. Standard and exotic singularities in cosmology.

WMAP + SDSS + Supernovae **combined bound on the dark energy barotropic index** w (Tegmark et al. (2004)); recent: e.g. Amanullah et al. (2010):



- **showed that there was no sharp cut-off of the data at $p = -\varrho$!!!** so that
- **the dark energy with $p < -\varrho$ (phantom) could be admitted!**
- **cosmic "no-hair" theorem violation - even a small fraction of phantom dark energy may dominate the evolution - Big-Rip singularity**
- NEC, WEC, DEC violated!

Big-Rip (type I) as an exotic (neither BB nor BC) singularity.

Since for phantom $w < -1$, then for convenience we may take

$$| w + 1 | = -(w + 1) > 0 , \tag{5}$$

so $a(t) = t^{-2/3|w+1|}$ and the conservation law for phantom gives

$$\varrho \propto a^{3|w+1|} . \tag{6}$$

- Conclusion: the bigger the universe grows, the denser it is, and **it becomes dominated by phantom (which overcomes Λ-term)** – an exotic future singularity appears – Big-Rip $\varrho, p \to \infty$ for $a \to \infty$
- Curvature invariants R^2, $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ diverge at Big-Rip
- In a Big-Rip scenario everything is pulled apart on the approach to a Big-Rip in a reverse order (Caldwell et al. PRL '03). Specifically, for $w = -3/2$ Big-Rip will happen in 20 Gyr.

Sudden Future Singularity (type II) as an exotic singularity.

Observational support for a Big-Rip gave a push to studies some other exotic types of singularities as possible sources of dark energy

Barrow (2004) dropped an assumption about the imposition of the equation of state (3)

$$p \neq p(\varrho), \tag{7}$$

and investigated how the energy density and pressure evolves if one assumes the analytic form of the scale factor only:

$$a(t) = a_s \left[\delta + (1-\delta)\, y^m - \delta\, (1-y)^n\right] , \qquad y \equiv \frac{t}{t_s} \tag{8}$$

where $a_s \equiv a(t_s) =$ const. and $\delta, m, n =$ const.

$$\dot{a} = a_s \left[\frac{m}{t_s}\,(1-\delta)\, y^{m-1} + \delta \frac{n}{t_s}\,(1-y)^{n-1}\right] , \tag{9}$$

$$\ddot{a} = \frac{a_s}{t_s^2} \left[m\,(m-1)\,(1-\delta)\, y^{m-2} - \delta n\,(n-1)\,(1-y)^{n-2}\right] . \tag{10}$$

Sudden Future Singularity ...

Provided

$$1 < n < 2, \tag{11}$$

one gets apart from a Big-Bang at $t = 0$ there is a new type of singularity at $t = t_s$ - a Sudden Future Singularity (SFS) (or type II - Nojiri, Odintsov, Tsujikawa 2005) which:

– manifests as a singularity of pressure p (or $\ddot{a}$) only

– leads to the dominant energy condition violation only In fact we have:

$$\begin{aligned} a = \text{const.}, \quad \dot{a} = \text{const.} \quad \varrho = \text{const.} \\ \ddot{a} \to -\infty \quad p \to \infty \quad \text{for} \quad t \to t_s \end{aligned} \tag{12}$$

Interesting point:

Schwarzschild horizon at $r = r_g$ - metric singular, curvature invariants regular,

Sudden Future Singularity at $t = t_s$ - metric regular, curvature invariants diverge.

Generalized Sudden Future singularities (type IIg).

Sudden future singularities may be generalized to GSFS if we take a general scale factor time derivative of an order r:

$$a^{(r)} = a_s \left[\frac{m(m-1)...(m-r+1)}{t_s^r} (1-\delta)\, y^{m-r} + (-1)^{r-1} \delta \frac{n(n-1)...(n-r+1)}{t_s^r} (1-y)^{n-r} \right], \qquad (13)$$

and choose (Barrow 2004, Lake 2004) $r-1<n<r$. Then for any integer r we have a singularity in the scale factor derivative $a^{(r)}$, and consequently in the appropriate pressure derivative $p^{(r-2)}$.

None of the energy conditions (EC) are violated for $r \geq 3$!!!

Finite Scale Factor (type III) and Big Separation (type IV).

The new exotic singularities were found by Type III singularities which we will call Finite Scale Factor - FSF singularities are characterized by the following conditions (Nojiri, Odintsov, Tsujikawa 2005):

$$a = a_s = \text{const.}, \; \varrho, \dot{a}_s \to \infty, \; |p|, \ddot{a}_s \to \infty$$

The simplest way to get them is to apply the scale factor as given previously for SFS, i.e.,

$$a(t) = a_s \left[\delta + (1-\delta)\, y^m - \delta\, (1-y)^n\right] , \qquad y \equiv \frac{t}{t_s} \tag{14}$$

where $a_f \equiv a(t_f) = \text{const.}$ and $\delta, A, m, n = \text{const.}$, but with the range of parameter n changed from $1 < n < 2$ onto

$$0 < n < 1$$

Big Separation - BS (type IV)

Type IV singularity is when (Nojiri, Odintsov, Tsujikwawa):

$$a = a_s = \text{const.},\ \varrho \to 0,\ p \to 0,\ \dddot{a},\ \ddot{H} \to \infty \text{ etc.}$$

and so it is similar to Generalized Sudden Future singularity with only one exception: it also gives the divergence of the barotropic index in the barotropic equation of state

$$p(t) = w(t)\varrho(t)$$

i.e.,

$$w(t) \to \infty$$

Barotropic index $w-$singularity (Type V)

Assuming the following type of scale factor (MPD, Denkiewicz 2009):

$$\begin{aligned} a(t) &= \frac{a_s}{1-\frac{3\gamma}{2}\left(\frac{n-1}{n-\frac{2}{3\gamma}}\right)^{n-1}} \\ &+ \frac{1-\frac{2}{3\gamma}}{n-\frac{2}{3\gamma}}\frac{na_s}{1-\frac{2}{3\gamma}\left(\frac{n-\frac{2}{3\gamma}}{n-1}\right)^{n-1}}\left(\frac{t}{t_s}\right)^{\frac{2}{3\gamma}} \\ &+ \frac{a_s}{\frac{3\gamma}{2}\left(\frac{n-1}{n-\frac{2}{3\gamma}}\right)^{n-1}-1}\left(1-\frac{1-\frac{2}{3\gamma}}{n-\frac{2}{3\gamma}}\frac{t}{t_s}\right)^n, \end{aligned} \tag{15}$$

with the admissible values of the parameters: $\gamma = w + 1 > 0$ and $n \neq 1$.

$w-$singularity

one gets a blow-up of the effective barotropic index, i.e.,

$$w(t_s) = \frac{1}{3}\left[2q(t_s) - 1\right] \to \infty \ , \tag{16}$$

accompanied by

$$p(t_s) \to 0; \quad \varrho(t_s) \to 0 \ . \tag{17}$$

There is an amazing duality between the Big-Bang and the w-singularity in the form

$$p_{BB} \leftrightarrow \frac{1}{p_w}, \quad \varrho_{BB} \leftrightarrow \frac{1}{\varrho_w}, \quad w_{BB} \leftrightarrow \frac{1}{w_w} \ . \tag{18}$$

In other words:

$$p_{BB} \to \infty; \varrho_{BB} \to \infty; w_{BB} \to 0; a_{BB} \to 0$$

$$p_w \to 0; \varrho_w \to 0; w_w \to \infty; a_w \to a_s = \text{const.}$$

Classification of exotic sing. (Nojiri et al. 2005, MPD & Denkiewicz 2010).

- Type 0 - Big-Bang (Big-Crunch) $a \to 0$, $p \to \infty$, $\varrho \to \infty$
- Type I - Big-Rip $a(t_s) \to \infty$ ($t_s < \infty$), $p \to \infty$, $\varrho \to \infty$ (Caldwell 2002)
- Type II - Sudden Future (includes Big Boost and Big-Brake) $a(t_s) =$ const., $\varrho =$ const., $p \to \infty$ (Barrow 2004)
- Type IIg - Generalized Sudden Future $a(t_s)$= const., $\varrho =$ const., $p =$const., $\dddot{a} \to \infty$ etc., $w < \infty$ (Barrow 2004)
- Type III - Finite Scale Factor (also Big-Freeze) $a(t_s)$ = const., $\varrho \to \infty$, $p \to \infty$ (NOT 2005, Denkiewicz 2011)
- Type IV - Big Separation: $a(t_s)$= const., $p = \varrho = 0$, $w \to \infty$, $\dddot{a} \to \infty$ etc. (NOT 2005) (and generalizations $p = \varrho =$const. Yurov 2010)
- Type V - w-singularity $a(t_s)$= const., $p = \varrho = 0$, $w \to \infty$ (MPD, Denkiewicz 2009) (and generalizations $p =$const. Yurov 2010)
- Little-Rip $a(t_s) \to \infty$, $\varrho(t_s) \to \infty$ ($t_s \to \infty$),
- Pseudo-Rip $\varrho(t_s) < \infty$ ($t_s \to \infty$) (Frampton et al. 2011, 2012)

Are these really singularities - strength?

As an example let us take an SFS which is determined by a blow-up of the Riemann tensor and its derivatives.

Geodesics do not feel SFSs at all, since geodesic equations are not singular for $a_s = a(t_s) =$ const. (Fernandez-Jambrina, Lazkoz PRD 74, 064030 (2006))

$$\left(\frac{dt}{d\tau}\right)^2 = A + \frac{P^2 + KL^2}{a^2(t)} , \tag{19}$$

$$\frac{dr}{d\tau} = \frac{P_1 cos\phi + P_2 \sin\phi}{a^2(t)} \sqrt{1 - Kr^2} , \tag{20}$$

$$\frac{d\phi}{d\tau} = \frac{L}{a^2(t) r^2} . \tag{21}$$

Geodesic deviation equation

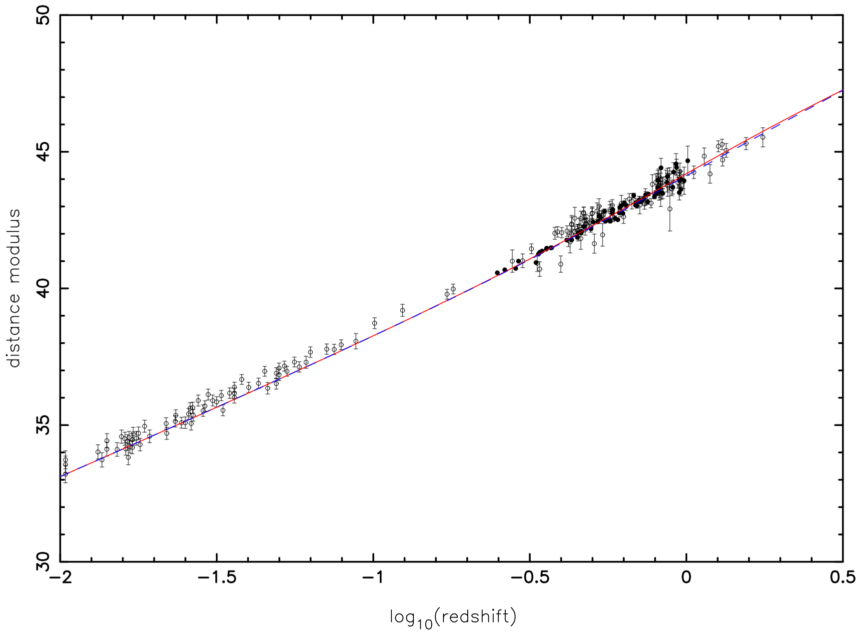
$$\frac{D^2 n^\alpha}{d\lambda^2} + R^{\alpha}_{\beta\gamma\delta} u^\beta n^\gamma u^\delta = 0 , \tag{22}$$

feels SFS since at $t = t_s$ we have the Riemann tensor $R^{\alpha}_{\beta\gamma\delta} \to \infty$.

Classification of exotic singularities - strength.

- Tipler's (Phys. Lett. A64, 8 (1977)) definition (of a weak singularity):
  $\int_0^\tau d\tau' \int_0^{\tau'} d\tau'' R_{ab} u^a u^b$
  does not diverge on the approach to a singularity at $\tau = \tau_s$
- Królak's (CQG 3, 267 (1988)) definition (of a weak singularity):
  $\int_0^\tau d\tau' R_{ab} u^a u^b$
  does not diverge on the approach to a singularity at $\tau = \tau_s$
- Type 0 (BB, BC): T, K - strong
- Type I (BR): T, K - strong
- Type II (SFS): T, K - weak
- Type IIg (GSFS): T, K - weak
- Type III (FSF): T - weak, K - strong
- Type IV (BS): T, K - weak
- Type V (w-sing.): T, K - weak (Fernandez-Jambrina (PRD, 2010)

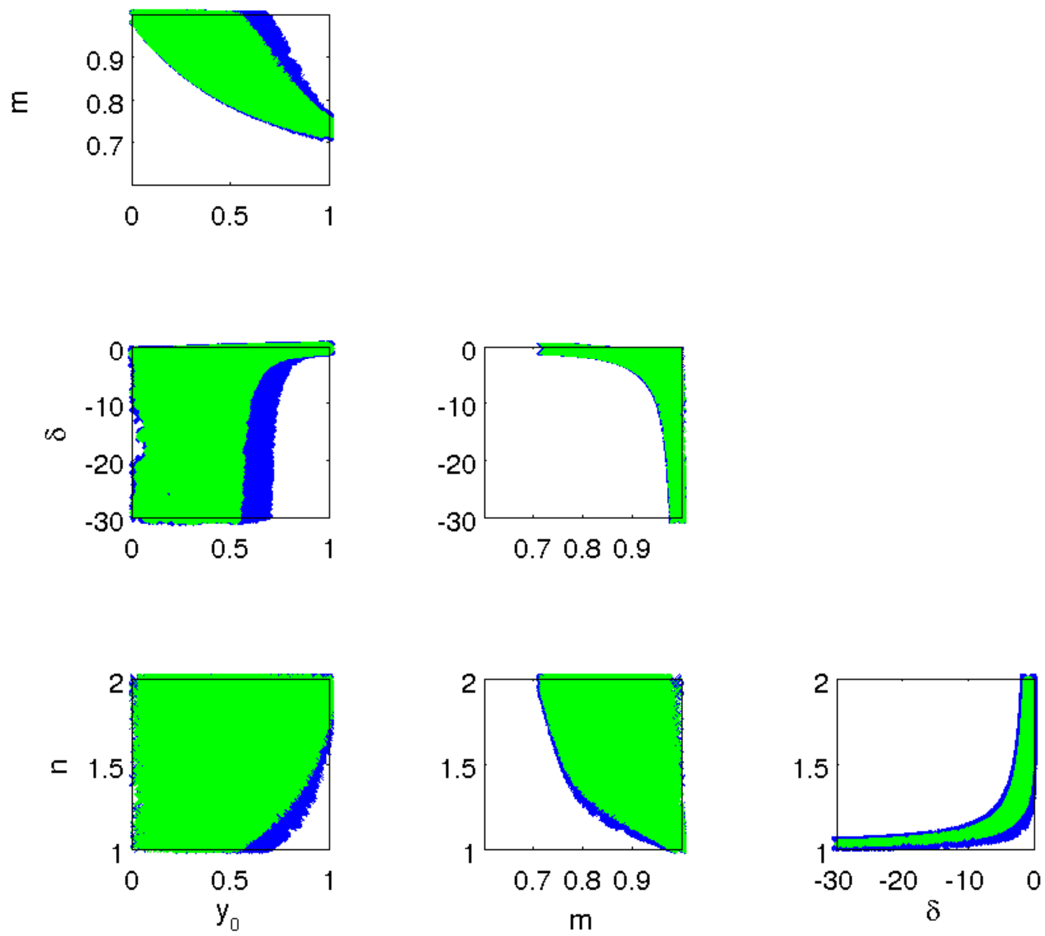
Exotic singularities can mimic dark energy.



SFS - supernovae only (MPD et al. 2007): distance modulus $\mu_L = m - M$ for the CC model ($H_0 = 72$kms^{-1}Mpc^{-1}, $\Omega_{m0} = 0.26$, $\Omega_{\Lambda 0} = 0.74$) (dashed curve) and SFS model ($m = 2/3 = 0.6666, n = 1.9999, \delta = -0.471, y_0 = 0.99936$) (solid curve). Open circles are for the 'Gold' data and filled circles are for SNLS data.

SFS: Supernovae, CMB shift parameter, and BAO (Denkiewicz et al. 2012)-

fits if $m \approx 0.72$, $w = -0.82$. Also FSF can do (Denkiewicz 2012).



3. Varying constants theories.

- It has been shown that quantum effects (e.g. Houndjo et al. arXiv:1203.6084) may change the strength of exotic singularities (e.g SFS to FSF).
- On the other hand, varying constants cosmologies have been applied to solve standard cosmology problems such as the horizon and flatness problem (e.g. Albrecht, Magueijo 1999).
- Our idea is to apply them to solve the singularity problem in cosmology.
- We can also ask if varying constants theories can soften/strengthen the standard and exotic singularities?

varying constants theories

First fully quantitative framework: Brans-Dicke scalar-tensor gravity (1961)

The gravitational constant G is associated with an average gravitational potential (scalar field) ϕ surrounding a given particle:

$< \phi >= GM/(c/H_0) \propto 1/G = 1.35 \times 10^{28} g/cm$. The scalar field gives the strength of gravity

$$G = \frac{1}{16\pi\Phi} \tag{23}$$

With the action

$$S = \int d^4x\sqrt{-g}\left(\Phi R - \frac{\omega}{\Phi}\partial_\mu\Phi\partial^\mu\Phi + \Lambda + L_m\right) \tag{24}$$

it relates to low-energy-effective superstring theory for $\omega = -1$

String coupling constant (running) $g_s = \exp(\phi/2)$ changes in time with ϕ - the dilaton and $\Phi = \exp(-\phi)$.

varying constants theories

Varying speed of light theories (VSL): Albrecht & Magueijo model (AM model) (1999)(Barrow 1999; Magueijo 2003):

$$c^4 = \psi(x^\mu) \tag{25}$$

and so the action is

$$S = \int d^4x \sqrt{-g} \left[\frac{\psi(R + 2\Lambda)}{16\pi G} + L_m + L_\psi \right] \tag{26}$$

AM model breaks Lorentz invariance (relativity principle and light principle) - preferred frame (cosmological or CMB) in which the field is minimally coupled to gravity.

Solves basic problems of standard cosmology: horizon problem and flatness problem.

Ansatz: Friedmann with $\rho = \rho_0 a^{-3\gamma}$, $c(t) = c_0 a^n$ - solution if $n \leq (1/2)(2 - 3\gamma)$.

varying constants theories

Magueijo covariant (conformally) and locally invariant model (2000, 2001):

$$\psi = \ln\left(\frac{c}{c_0}\right) \quad \text{or} \quad c = c_0 e^{\psi} \,, \tag{27}$$

with the action

$$S = \int d^4x \sqrt{-g} \left[\frac{c_0^4 e^{\alpha\psi} (R + 2\Lambda + L_\psi)}{16\pi G} + e^{\beta\psi} L_m \right] \,, \tag{28}$$

with

$$L_\psi = \kappa(\psi) \nabla_\mu \psi \nabla^\mu \psi \,. \tag{29}$$

Further assumption: $\alpha - \beta = 4$.

Interesting subcases:

$\alpha = 4; \beta = 0$ - Brans-Dicke with $\phi_{BD} = e^{4\psi}/G$ and $\kappa(\psi) = 16\omega_{BD}(\phi_{BD})$.

$\alpha = 0; \beta = -4$ - minimal VSL theory.

varying constants theories

Varying fine structure constant α (or charge $e = e_0\epsilon(x^\mu)$ theories (Webb et al. 1999, Sandvik 2002)

$$S = \int d^4x\sqrt{-g}\left(\psi R - \frac{\omega}{2}\partial_\mu\psi\partial^\mu\psi - \frac{1}{4}f_{\mu\nu}f^{\mu\nu}e^{-2\psi} + L_m\right) \tag{30}$$

with $\psi = \ln\epsilon$ and $f_{\mu\nu} = \epsilon F_{\mu\nu}$.
Assume linear expansion $e^\psi = 1 - 8\pi G\zeta(\psi - \psi_0) = 1 - \Delta\alpha/\alpha$ with the constraint on the local equivalence principle violence $\mid\zeta\mid\leq 10^{-3}$. The relation to DE is:

$$\gamma = \frac{(8\pi G\frac{d\psi}{d\ln a})^2}{\Omega_\psi} \tag{31}$$

This can be tested while mimicking the dark energy by spectrograph CODEX (COsmic Dynamics EXplorer) a device attached to planned E-ELT (European Extremely Large Telescope) measuring the so-called redshift drift (or Sandage-Loeb effect) for $2 < z < 5$ (Vielzeuf and Martins 2012).

4. Varying constants versus cosmic singularities.

We consider the Friedmann universes in varying speed of light (VSL) theories and varying gravitational constant G theories as follows (ϱ - mass density; $\varepsilon = \varrho c^2(t)$ - energy density in $Jm^{-3} = Nm^{-2} = kgm^{-1}s^{-2}$)

$$\varrho(t) = \frac{3}{8\pi G(t)}\left(\frac{\dot{a}^2}{a^2} + \frac{kc^2(t)}{a^2}\right) , \tag{32}$$

$$p(t) = -\frac{c^2(t)}{8\pi G(t)}\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{kc^2(t)}{a^2}\right) , \tag{33}$$

and the energy-momentum “conservation law” is

$$\dot{\varrho}(t) + 3\frac{\dot{a}}{a}\left(\varrho(t) + \frac{p(t)}{c^2(t)}\right) = -\varrho(t)\frac{\dot{G}(t)}{G(t)} + 3\frac{kc(t)\dot{c}(t)}{4\pi G a^2} . \tag{34}$$

General form of the scale factor.

We suggest a general form of the scale factor (MPD, K. Marosek, JCAP 02 (2013), 012), which admits big-bang, big-rip, sudden future, finite scale factor and w-singularities and reads as

$$a(t) = a_s \left(\frac{t}{t_s}\right)^m \exp\left(1 - \frac{t}{t_s}\right)^n , \qquad (35)$$

with the constants t_s, a_s, m, n. For $k = 0$ we have

$$\varrho(t) = \frac{3}{8\pi G(t)} \left[\frac{m}{t} - \frac{n}{t_s}\left(1 - \frac{t}{t_s}\right)^{n-1}\right]^2 , \qquad (36)$$

$$p(t) = -\frac{c^2(t)}{8\pi G(t)} \left[\frac{m(3m-2)}{t^2} - 6\frac{mn}{tt_s}\left(1 - \frac{t}{t_s}\right)^{n-1} + 3\frac{n^2}{t_s^2}\left(1 - \frac{t}{t_s}\right)^{2(n-1)} + 2\frac{n(n-1)}{t_s^2}\left(1 - \frac{t}{t_s}\right)^{n-2}\right] . \qquad (37)$$

The scale factor.

For $m < 0$ we have a big-rip singularity - $a \to \infty$, $\varrho \to \infty$, $p \to \infty$ at $t = 0$;
For $1 < n < 2$ we have a sudden future singularity (SFS) which appears at $t = t_s$ ($a = a_s$, $\varrho =$ const., $p \to \infty$);
For $0 < n < 1$ we have a stronger finite scale factor singularity (FSF) at $t = t_s$ ($a = a_s$, $\varrho \to \infty$, $p \to \infty$).
In fact, for $1 < n < 2$ only the last term in the pressure of the type $(1 - t/t_s)^{n-2}$ blows-up, while for $0 < n < 1$ two more terms $(1 - t/t_s)^{n-1}$ and $(1 - t/t_s)^{2(n-1)}$ do.

Regularizing singularities by varying constants

One bears in mind the scale factor (35), the energy density (36) and pressure (37)

Regularizing a Big-Bang singularity by varying G:

If

$$G(t) \propto \frac{1}{t^2} \tag{38}$$

which is a faster decrease than in Dirac's LNH $G \propto 1/t$, but influences less the temperature of the Earth constraint (Teller 1948).

Both divergence in ϱ and p are removed, though at the expense of having the "singularity" of strong gravitational coupling $G \to \infty$ at $t \to 0$.

In the Dirac's case, only the ϱ singularity can be removed.

regularizing singularities by varying constants: SFS

Regularizing an SFS singularity by varying c:

If

$$c(t) = c_0 \left(1 - \frac{t}{t_s}\right)^{\frac{p}{2}} , \tag{39}$$

then

$$p(t) = -\frac{c_0^2}{8\pi G} \left[\frac{m(3m-2)}{t^2} \left(1 - \frac{t}{t_s}\right)^{p} \right.$$

$$\left. - \quad 6\frac{mn}{tt_s} \left(1 - \frac{t}{t_s}\right)^{p+n-1} + 3\frac{n^2}{t_s^2} \left(1 - \frac{t}{t_s}\right)^{p+2n-2} + 2\frac{n(n-1)}{t_s^2} \left(1 - \frac{t}{t_s}\right)^{p+n-2} \right] .$$

and the singularity of pressure is regularized provided $p > 2 - n$, $(1 < n < 2)$.

regularizing singularities by varying constants: SFS.

Physical consequence: light eventually stops at the singularity. Same happens in loop quantum cosmology (LQC) where it is called the anti-newtonian limit $c = c_0\sqrt{1-\varrho/\varrho_c} \to 0$ for $\varrho \to \varrho_c$ with ϱ_c being the critical density (Cailettau et al. 2012). The low-energy limit $\varrho \ll \varrho_0$ gives the standard limit $c \to c_0$.

It also appears naturally in Magueijo model ((Magueijo, PRD 63, 043502 (2001))) in which black holes are not reachable since the light stops at the horizon (despite they possess Schwarzschild singularity).

Besides, both options $c = 0$ and $c = \infty$ are possible in this model.

regularizing singularities by varying constants: w-sing.

In the limit $m \to 0$ we have an exotic singularity scale factor given by $a(t) = a_s \exp\left(1 - t/t_s\right)$ and so from (36) and (37) we have

$$\varrho_{ex}(t) = \frac{3}{8\pi G(t)} \frac{n^2}{t_s^2} \left(1 - \frac{t}{t_s}\right)^{2(n-1)} , \quad (40)$$

$$p_{ex}(t) = -\frac{c^2(t)}{8\pi G(t)} \left[3\frac{n^2}{t_s^2} \left(1 - \frac{t}{t_s}\right)^{2(n-1)} + 2\frac{n(n-1)}{t_s^2} \left(1 - \frac{t}{t_s}\right)^{n-2}\right] \quad (41)$$

so that

$$w_{ex}(t) = \frac{p_{ex}(t)}{\varepsilon_{ex}(t)} = -\left[1 + \frac{2}{3}\frac{n-1}{n}\frac{1}{\left(1 - \frac{t}{t_s}\right)^n}\right] = -\left[\frac{1}{3} - \frac{2}{3}q_{ex}(t)\right] , \quad (42)$$

which is a w-singularity for $n > 2$ ($p = \varrho = 0$, $w_{ex} \to \infty$). Its regularization by varying $c(t)$ is impossible since there is no c-dependence here.

regularizing singularities by varying constants: SFS

Regularizing an SFS singularity by varying G:

If we assume that

$$G(t) = G_0 \left(1 - \frac{t}{t_s}\right)^{-r} , \tag{43}$$

($r =$ const., $G_0 =$ const.) which changes (36) and (37) to

$$\begin{aligned}\varrho(t) &= \frac{3}{8\pi G_0}\left[\frac{m^2}{t^2}\left(1-\frac{t}{t_s}\right)^{r} - \frac{2mn}{tt_s}\left(1-\frac{t}{t_s}\right)^{r+n-1}\right. \\ &+ \left.\frac{n^2}{t_s^2}\left(1-\frac{t}{t_s}\right)^{r+2n-2}\right] , \end{aligned} \tag{44}$$

$$\begin{aligned}p(t) &= -\frac{c^2}{8\pi G_0}\left[\frac{m(3m-2)}{t^2}\left(1-\frac{t}{t_s}\right)^{r} - 6\frac{mn}{tt_s}\left(1-\frac{t}{t_s}\right)^{r+n-1} + 3\frac{n^2}{t_s^2}\left(1-\frac{t}{t_s}\right)^{r+2n-2}\right. \\ &+ \left. 2\frac{n(n-1)}{t_s^2}\left(1-\frac{t}{t_s}\right)^{r+n-2}\right] . \end{aligned} \tag{45}$$

regularizing singularities by varying constants: SFS

From (44) and (45) it follows that an SFS singularity $(1 < n < 2)$ is regularized by varying gravitational constant when

$$r > 2 - n \ , \tag{46}$$

and an FSF singularity $(0 < 1 < n)$ is regularized when

$$r > 1 - n \ . \tag{47}$$

On the other hand, assuming that we have an SFS singularity and that

$$-1 < r < 0 \, , \tag{48}$$

we get that varying G may change an SFS singularity onto a stronger FSF singularity when

$$0 < r + n < 1 \, . \tag{49}$$

Regularizing singularities: (anti-)Chaplygin gas

The equation of state of the (anti-)Chaplygin gas reads as

$$p(t) = \pm \frac{A}{\varepsilon(t)} = \pm \frac{A}{\varrho(t) c^2(t)} \qquad (A > 0) \ , \tag{50}$$

where the "-" sign is for Chaplygin gas while the "+" sign is for anti-Chaplygin gas case and the unit of A is the energy density(=pressure) square $J^2 m^{-6}$. Inserting (50) into (34) gives

$$\dot{\varrho}(t) + 3\frac{\dot{a}}{a}\left(\frac{\varrho^2 c^4(t) \mp A}{\varrho(t) c^4(t)}\right) = -\varrho(t)\frac{\dot{G}(t)}{G(t)} + 3\frac{kc(t)\dot{c}(t)}{4\pi G(t) a^2} \ . \tag{51}$$

We assume both varying $G = G(t)$ and $c = c(t)$ though with zero curvature $(k = 0)$ as follows

$$\varrho(t) c^2(t) = B = \text{const.} \ , \tag{52}$$

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The solution of (51) reads as

$$\varrho(t)a^{3\gamma}(t)G(t) = E = \text{const.}\ , \tag{53}$$

where we have defined

$$\gamma \equiv \frac{B^2 \mp A}{B^2} \tag{54}$$

Putting the standard big-bang scale factor $a(t) = (t/t_s)^{2/3\gamma}$, we finally have

$$\varrho(t) = \frac{Et_s^2}{t^2 G(t)}\ , \qquad p(t) = \mp\frac{A}{B} = \text{const.}\ , \tag{55}$$

which give $\varrho \to \infty$ and $p(0) = 0$ provided $G(0) = \text{const.} \neq 0$. The singularity at $t = 0$ in ϱ and p can be regularized by taking $G(t) \propto 1/t^2$ at the expense of having a constant pressure (cosmological term) instead of zero pressure.

Subtleties:

- In order to regularize an SFS or an FSF singularity by varying $c(t)$, the light should slow and eventually stop propagating at a singularity. Similar effects were found in loop quantum cosmology (LQC) as well as in VSL for Schwarzschild horizon (Magueijo 2001) - speed of light is either zero or infinity at $r = r_s$. An observer cannot reach this surface even in his finite proper time.
- To regularize an SFS, FSF by varying gravitational constant $G(t)$ - the strength of gravity has to become infinite at a singularity. On the one hand, it is quite reasonable because of the requirement to overcome an infinite (anti-)tidal forces at the singularity, but on the other hand, it makes another singularity - a singularity of strong coupling for a physical field such as $G \propto 1/\Phi$. Such problems were already dealt with in superstring and brane cosmology where both the curvature singularity and a strong coupling singularity appeared (choice of coupling, quantum corrections).

5. Conclusions

- Currently one is able to differentiate quite a number of cosmological singularities with completely different properties - despite many of them are geodesically complete, they still lead to a blow-up of various physical quantities (scale factor, energy density, pressure, physical fields).
- Some of these singularities may serve as dark energy, especially if they are quite close in the near future. For example, an SFS may even appear in 8.7 Myr with no contradiction with bare supernovae data. It can be fitted to a combined SnIa, CMB and BAO data, but at the expense of admitting an approach to a Big-Bang by a fluid which is not exactly dust (m=0.66), but has a slightly negative pressure (m = 0.73 and so w = -0.09).
- An interesting proposal is to investigate how the singularities are influenced by varying physical constants. In particular, we may look for the answer if it is possible to "regularize" (remove infinities) or change these singularities and what are the physical consequences of such an action, because what we face is usually the new "singularity" in a physical constant/field which acts to remove/change the type of singularity.