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# Weak singularities in Friedmann cosmology and varying physical constants

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## References

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- A. Balcerzak and MPD, PRD **73**, 101301(R) (2006) (hep-th/0604034)
- MPD, T. Denkiewicz, M.A. Hendry, PRD **75**, 123524 (2007)
- MPD, T. Denkiewicz, PRD **79**, 063521 (2009)
- MPD, T. Denkiewicz, 0910.0023 AIP Conference Proceedings **1241**, 561 (2010).
- MPD, PLB 702, 320 (2011).
- T. Denkiewicz, JCAP **07**, 036 (2012).
- MPD, K. Marosek, JCAP 02 (2013), 012 (arXiv: 1207.4038).
- MPD, K. Marosek, A. Balcerzak, ArXiv: 1308.5462 (Mem. della Societa Astronomica Italiana - to appear).
- T. Denkiewicz, MPD, C.J.A.P. Martins, P. Vielzeuf - 1402.0520, PRD to appear (2014).

## Plan:

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- 1. Introduction.
- 2. Standard and exotic singularities in cosmology.
- 3. The universe through an exotic singularity - averaging surprises.
- 4. Varying constants theories.
- 5. Varying constant versus cosmic singularities.
- 6. Conclusions.

# 1. Introduction.

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Standard Einstein-Friedmann equations are two equations for three unknown functions of time  $a(t), p(t), \rho(t)$

$$\rho = \frac{3}{8\pi G} \left( \frac{\dot{a}^2}{a^2} + \frac{K}{a^2} \right), \quad (1)$$

$$p = -\frac{1}{8\pi G} \left( 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2} \right). \quad (2)$$

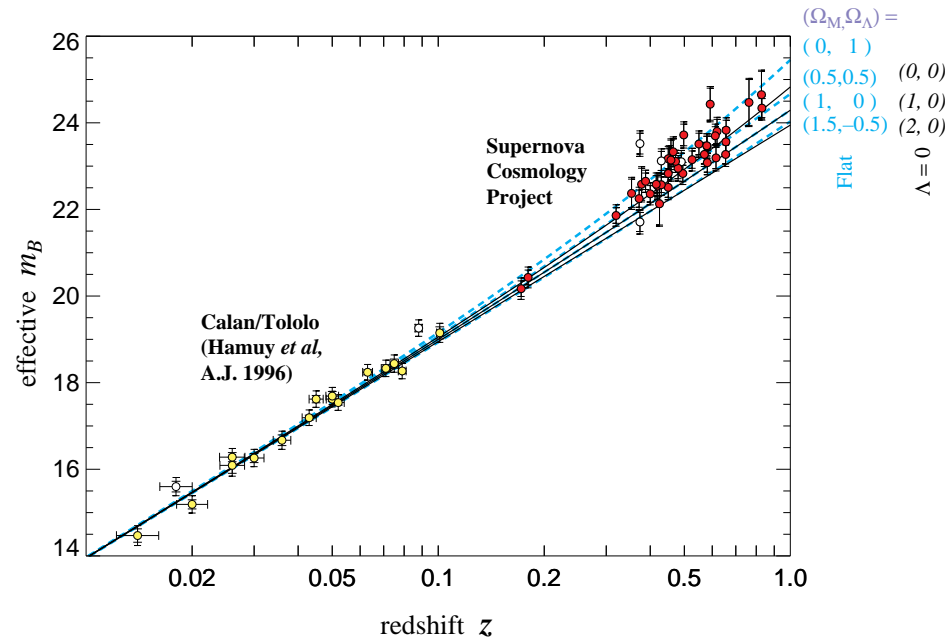
plus an equation of state, e.g., of a barotropic type ( $w = \text{const.} \geq -1$ ):

$$p(t) = w\rho(t) \quad \rightarrow \quad a(t) \propto t^{\frac{2}{3(w+1)}}. \quad (3)$$

Until very recently (including first supernovae results) most of cosmologists studied only simplest - say “standard” solutions - each of them starts with **Big-Bang** singularity in which  $a \rightarrow 0, \rho, p \rightarrow \infty$  – one of them (of  $K = +1$ ) terminates at the second singularity (**Big-Crunch**) where  $a \rightarrow 0, \rho, p \rightarrow \infty$  – the other two ( $K = 0, -1$ ) continue to an **asymptotic emptiness**  $\rho, p \rightarrow 0$  for  $a \rightarrow \infty$ .

BB and BC exhibit **geodesic incompleteness** and **curvature blow-up**.

## However, first supernovae observations ...



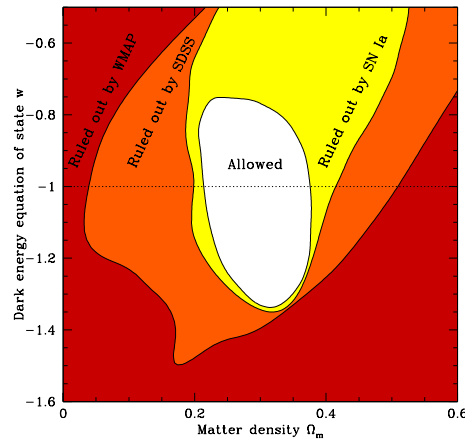
... gave evidence for the **strong** energy condition

$$\rho + 3p \geq 0, \quad \rho + p \geq 0. \quad (4)$$

violation, but **the paradigm of the “standard” Big-Bang/Crunch singularities remained untouched.**

## 2. Standard and exotic singularities in cosmology.

WMAP + SDSS + Supernovae **combined bound on the dark energy barotropic index**  $w$  (Tegmark et al. (2004)); recent: Shafer & Huterer 1312.1668



- **showed that there was no sharp cut-off of the data at  $p = -\rho!!!$  so that**
- **the dark energy with  $p < -\rho$  (phantom) could be admitted!**
- **cosmic “no-hair” theorem violation - even a small fraction of phantom dark energy may dominate the evolution - Big-Rip singularity**
- **NEC, WEC, DEC violated!**

## Big-Rip (type I) as an exotic (neither BB nor BC) singularity.

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Since for phantom  $w < -1$ , then for convenience we may take

$$|w + 1| = -(w + 1) > 0, \quad (5)$$

so  $a(t) = t^{-2/3|w+1|}$  and the conservation law for phantom gives

$$\rho \propto a^{3|w+1|}. \quad (6)$$

- Conclusion: **the bigger** the universe grows, **the denser** it is, and **it becomes dominated by phantom (which overcomes  $\Lambda$ -term) – an exotic future singularity appears – Big-Rip**  $\rho, p \rightarrow \infty$  for  $a \rightarrow \infty$
- Curvature invariants  $R^2, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$  **diverge** at Big-Rip
- In a Big-Rip scenario everything is **pulled apart** on the approach to a Big-Rip in a reverse order (Caldwell et al. PRL '03). Specifically, for  $w = -3/2$  Big-Rip will happen in 20 Gyr.

## Sudden Future Singularity (type II) as an exotic singularity.

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**Observational support for a Big-Rip gave a push to studies some other exotic types of singularities as possible sources of dark energy**

Barrow (2004) dropped an assumption about the imposition of the equation of state (3)

$$p \neq p(\rho), \quad (7)$$

and investigated how the energy density and pressure evolves if one assumes the analytic form of the scale factor only:

$$a(t) = a_s [\delta + (1 - \delta) y^m - \delta (1 - y)^n] , \quad y \equiv \frac{t}{t_s} \quad (8)$$

where  $a_s \equiv a(t_s) = \text{const.}$  and  $\delta, m, n = \text{const.}$

$$\dot{a} = a_s \left[ \frac{m}{t_s} (1 - \delta) y^{m-1} + \delta \frac{n}{t_s} (1 - y)^{n-1} \right] , \quad (9)$$

$$\ddot{a} = \frac{a_s}{t_s^2} \left[ m(m-1)(1-\delta)y^{m-2} - \delta n(n-1)(1-y)^{n-2} \right] . \quad (10)$$



## Sudden Future Singularity ...

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Provided

$$1 < n < 2, \quad (11)$$

one gets apart from a Big-Bang at  $t = 0$  there is a new type of singularity at  $t = t_s$   
- a **Sudden Future Singularity (SFS)** (or type II - Nojiri, Odintsov, Tsujikawa 2005) which:

- manifests as a singularity of pressure  $p$  (or  $\ddot{a}$ ) only
- leads to the dominant energy condition violation only. In fact we have:

$$\begin{aligned} a = \text{const.}, \quad \dot{a} = \text{const.}, \quad \rho = \text{const.} \\ \ddot{a} \rightarrow -\infty \quad p \rightarrow \infty \quad \text{for} \quad t \rightarrow t_s \end{aligned} \quad (12)$$

Interesting point:

**Schwarzschild** horizon at  $r = r_g$  - **metric singular**, **curvature invariants regular**,

**Sudden Future Singularity** at  $t = t_s$  - **metric regular**, **curvature invariants**

**diverge**.

## Sudden Future Singularities ...

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**commonly appear in LQC** (Cailletau et al. PRL 101, 251302 (2008)) on the contrary to big-bang and big-rip which are avoided in LQC (e.g. Sami et al. gr-qc/0605113).

This is due to quadratic term in the Friedmann equation (e.g. Bojowald PRL '02, gr-qc/0601085) which simulates negative brane tension (extra timelike dimensions - e.g. Shtanov and Sahni PLB, 557 (2003), 1):

$$H^2 = \frac{1}{3m_{pl}^2} \left( \rho - \frac{\rho^2}{\rho_c} \right) - \frac{k}{a^2}, \quad (13)$$

where the critical density is

$$\rho_c \equiv \frac{\sqrt{3}}{16\pi\gamma^3 G^2 \hbar^2}, \quad (14)$$

and  $\gamma$  is the Barbero-Immirzi parameter ( $\gamma \approx 0.2375$ , see Meissner gr-qc/0407052).

## Generalized Sudden Future singularities (type IIg).

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Sudden future singularities may be generalized to GSFS if we take a general scale factor time derivative of an order  $r$ :

$$a^{(r)} = a_s \left[ \frac{m(m-1)\dots(m-r+1)}{t_s^r} (1-\delta) y^{m-r} + (-1)^{r-1} \delta \frac{n(n-1)\dots(n-r+1)}{t_s^r} (1-y)^{n-r} \right], \quad (15)$$

and choose (Barrow 2004, Lake 2004)  $r-1 < n < r$ . Then for any integer  $r$  we have a **singularity** in the scale factor derivative  $a^{(r)}$ , and consequently **in** the appropriate **pressure derivative**  $p^{(r-2)}$ .

**None of the energy conditions (EC) are violated for  $r \geq 3!!!$**

## Finite Scale Factor (type III) and Big Separation (type IV).

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The new exotic singularities were found as Type III singularities which we will call **Finite Scale Factor - FSF** singularities are characterized by the following conditions (Nojiri, Odintsov, Tsujikawa 2005):

$$a = a_s = \text{const.}, \rho, \dot{a}_s \rightarrow \infty, |p|, \ddot{a}_s \rightarrow \infty$$

The simplest way to get them is to apply the scale factor as given previously for SFS, i.e.,

$$a(t) = a_s [\delta + (1 - \delta) y^m - \delta (1 - y)^n] , \quad y \equiv \frac{t}{t_s} \quad (16)$$

where  $a_f \equiv a(t_f) = \text{const.}$  and  $\delta, A, m, n = \text{const.}$ , but with the range of parameter  $n$  changed from  $1 < n < 2$  onto

$$0 < n < 1$$

## Big Separation - BS (type IV)

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Type IV singularity is when (Nojiri, Odintsov, Tsujikawa 2005):

$$a = a_s = \text{const.}, \varrho \rightarrow 0, p \rightarrow 0, \ddot{a}, \ddot{H} \rightarrow \infty \text{ etc.}$$

and so it is **similar** to Generalized Sudden Future singularity with only **one exception**: it also gives the divergence of the barotropic index in the barotropic equation of state

$$p(t) = w(t)\varrho(t)$$

i.e.,

$$w(t) \rightarrow \infty$$

## Barotropic index $w$ –singularity (Type V)

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Assuming the following type of scale factor (MPD, Denkiewicz 2009):

$$\begin{aligned} a(t) &= \frac{a_s}{1 - \frac{3\gamma}{2} \left( \frac{n-1}{n-\frac{2}{3\gamma}} \right)^{n-1}} \\ &+ \frac{1 - \frac{2}{3\gamma}}{n - \frac{2}{3\gamma}} \frac{na_s}{1 - \frac{2}{3\gamma} \left( \frac{n-\frac{2}{3\gamma}}{n-1} \right)^{n-1}} \left( \frac{t}{t_s} \right)^{\frac{2}{3\gamma}} \\ &+ \frac{a_s}{\frac{3\gamma}{2} \left( \frac{n-1}{n-\frac{2}{3\gamma}} \right)^{n-1} - 1} \left( 1 - \frac{1 - \frac{2}{3\gamma}}{n - \frac{2}{3\gamma}} \frac{t}{t_s} \right)^n, \end{aligned} \quad (17)$$

with the admissible values of the parameters:  $\gamma = w + 1 > 0$  and  $n \neq 1$ .

## *w*–singularity

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one gets a blow-up of the effective barotropic index, i.e.,

$$w(t_s) = \frac{1}{3} [2q(t_s) - 1] \rightarrow \infty, \quad (18)$$

accompanied by

$$p(t_s) \rightarrow 0; \quad \varrho(t_s) \rightarrow 0. \quad (19)$$

There is a kind of **duality between the Big-Bang and the *w*-singularity** in the form

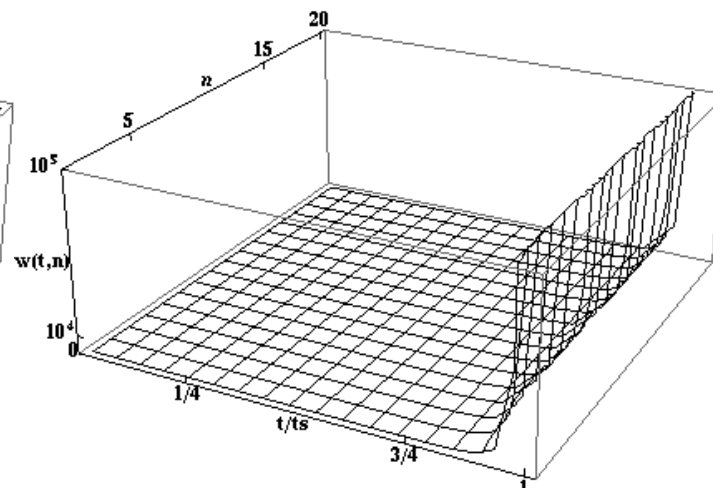
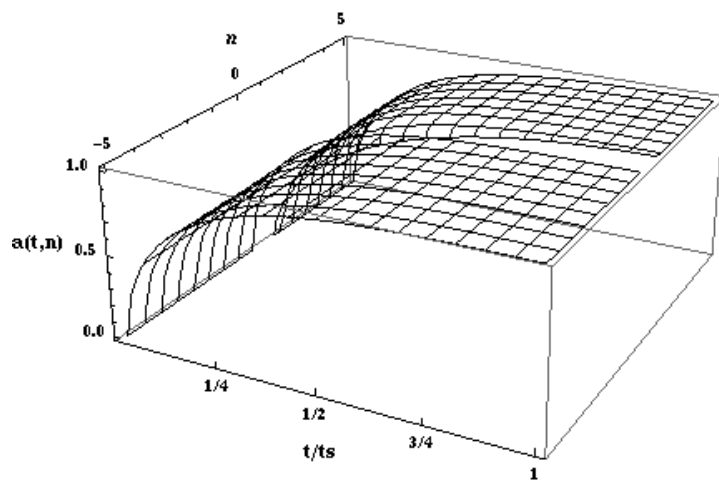
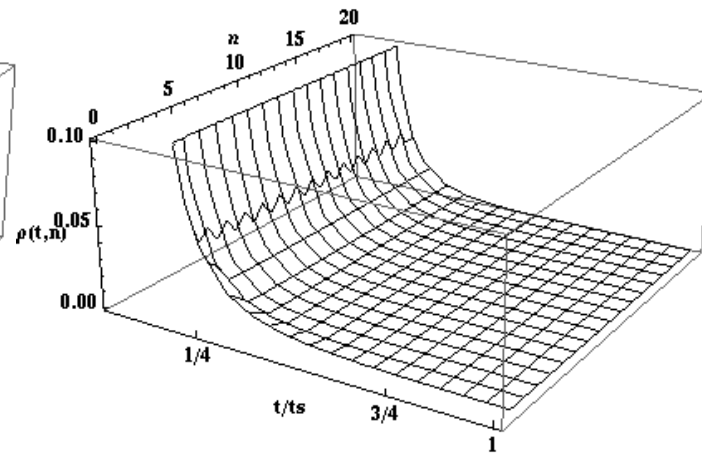
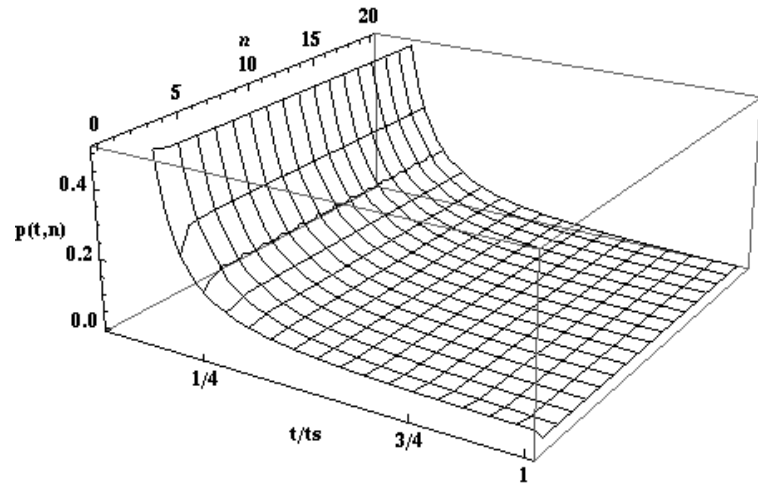
$$p_{BB} \leftrightarrow \frac{1}{p_w}, \quad \varrho_{BB} \leftrightarrow \frac{1}{\varrho_w}, \quad w_{BB} \leftrightarrow \frac{1}{w_w}. \quad (20)$$

In other words:

$$\begin{aligned} p_{BB} &\rightarrow \infty; \varrho_{BB} \rightarrow \infty; w_{BB} \rightarrow 0; a_{BB} \rightarrow 0 \\ p_w &\rightarrow 0; \varrho_w \rightarrow 0; w_w \rightarrow \infty; a_w \rightarrow a_s = \text{const.} \end{aligned}$$

*w*-sing. appear in:  $f(R)$  gravity (Starobinsky '80), scalar field gravity (Setare, Saridakis '09), and brane gravity (Sahni, Shtanov '05)

# $w$ -duality





## Classification of exotic sing. (Nojiri et al. 2005, MPD & Denkiewicz 2010).

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- Type 0 - Big-Bang (Big-Crunch)  $a \rightarrow 0, p \rightarrow \infty, \rho \rightarrow \infty$
- Type I - Big-Rip  $a(t_s) \rightarrow \infty (t_s < \infty), p \rightarrow \infty, \rho \rightarrow \infty$  (Caldwell 2002)
- Type II - Sudden Future (includes Big Boost and Big-Brake)  $a(t_s) = \text{const.}, \rho = \text{const.}, p \rightarrow \infty$  (Barrow 2004)
- Type IIg - Generalized Sudden Future  $a(t_s) = \text{const.}, \rho = \text{const.}, p = \text{const.}, \ddot{a} \rightarrow \infty$  etc.,  $w < \infty$  (Barrow 2004)
- Type III - Finite Scale Factor (also Big-Freeze)  $a(t_s) = \text{const.}, \rho \rightarrow \infty, p \rightarrow \infty$  (NOT 2005, Denkiewicz 2011)
- Type IV - Big Separation:  $a(t_s) = \text{const.}, p = \rho = 0, w \rightarrow \infty, \ddot{a} \rightarrow \infty$  etc. (NOT 2005) (and generalizations  $p = \rho = \text{const.}$  Yurov 2010)
- Type V -  $w$ -singularity  $a(t_s) = \text{const.}, p = \rho = 0, w \rightarrow \infty$  (MPD, Denkiewicz 2009) (and generalizations  $p = \text{const.}$  Yurov 2010)
- Little-Rip  $a(t_s) \rightarrow \infty, \rho(t_s) \rightarrow \infty (t_s \rightarrow \infty),$
- Pseudo-Rip  $\rho(t_s) < \infty (t_s \rightarrow \infty)$  (Frampton et al. 2011, 2012)

## Are these really singularities - strength?

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As an example let us take an SFS which is determined by a **blow-up of the Riemann tensor** and its derivatives.

Geodesics do not feel SFSs at all, since geodesic equations are not singular for  $a_s = a(t_s) = \text{const.}$  (Fernandez-Jambrina, Lazkoz PRD 74, 064030 (2006))

$$\left(\frac{dt}{d\tau}\right)^2 = A + \frac{P^2 + KL^2}{a^2(t)}, \quad (21)$$

$$\frac{dr}{d\tau} = \frac{P_1 \cos\phi + P_2 \sin\phi}{a^2(t)} \sqrt{1 - Kr^2}, \quad (22)$$

$$\frac{d\phi}{d\tau} = \frac{L}{a^2(t)r^2}. \quad (23)$$

Geodesic deviation equation

$$\frac{D^2 n^\alpha}{d\lambda^2} + R^\alpha_{\beta\gamma\delta} u^\beta n^\gamma u^\delta = 0, \quad (24)$$

feels SFS since at  $t = t_s$  we have the Riemann tensor  $R^\alpha_{\beta\gamma\delta} \rightarrow \infty$

## Classification of exotic singularities - strength.

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- Tipler's (Phys. Lett. A64, 8 (1977)) definition (of a strong singularity):

$$\int_0^\tau d\tau' \int_0^{\tau'} d\tau'' |R_{ajb}^i u^a u^b|$$

diverges on the approach to a singularity at  $\tau = \tau_s$

- i.e. an extended object is crushed to zero volume (represented by three linearly independent, vorticity-free geodesic deviation vectors at  $p$  along causal geodesic  $l$ ) at the singularity by infinite tidal forces

- Królak's (CQG 3, 267 (1988)) definition (of a strong singularity):

$$\int_0^\tau d\tau' |R_{ajb}^i u^a u^b|$$

diverges on the approach to a singularity at  $\tau = \tau_s$

- i.e. the expansion of every future-directed congruence of null (timelike) geodesics emanating from point  $p$  and containing  $l$  becomes negative somewhere on  $l$

## Classification of exotic singularities - strength.

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Fernandez-Jambrina (PRD 82, 124004 (2010)) used Puiseux series expansion

$$a(t) = c_0 + (t_s - t)^{\eta_0} + c_1 (t_s - t)^{\eta_1} + c_2 (t_s - t)^{\eta_2} + \dots \quad \eta_0 < \eta_1 < \dots \quad c_0 > 0 \quad (25)$$

to check the strength of exotic singularities (T - Tipler; K - Królak)

Balcerzak and MPD PRD'06) considered classical Polyakov strings

$$S = -\frac{T}{2} \int d\tau d\sigma \eta^{ab} g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \quad (26)$$

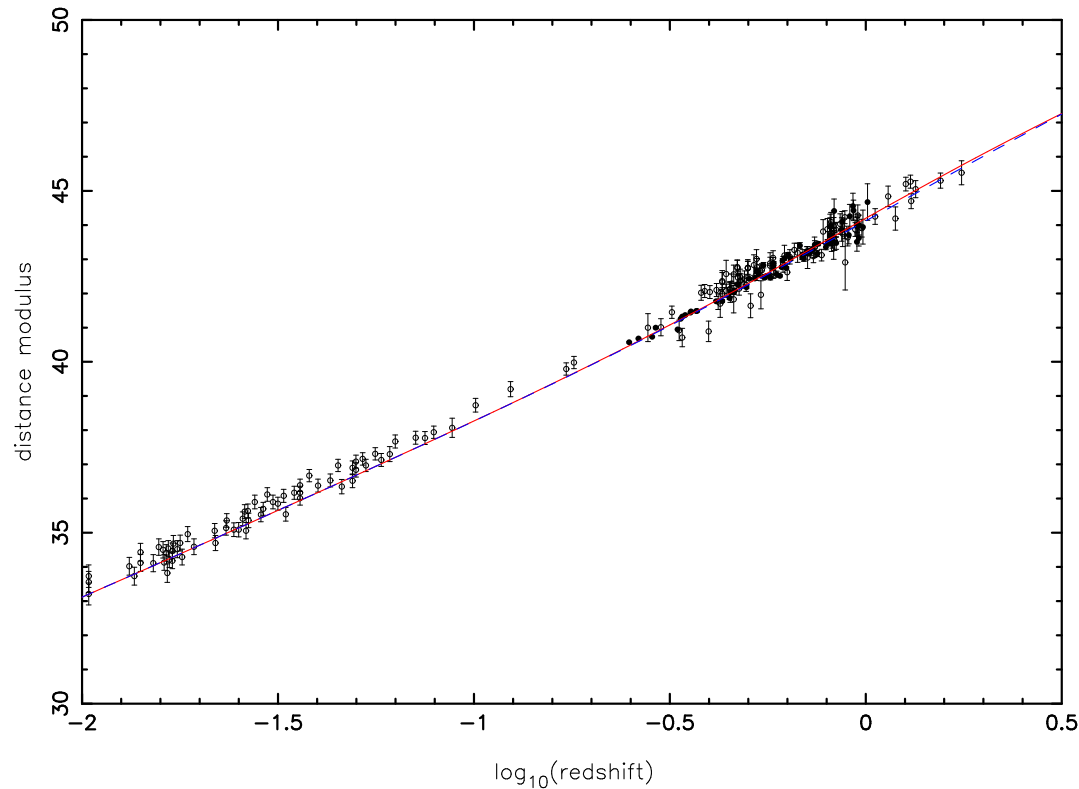
with an invariant size  $S(\tau) = 2\pi a(\eta(\tau))R(\tau)$  (circular ansatz with radius  $R$ ) falling into exotic singularities to show that they are: **infinitely stretched**  $S \rightarrow \infty$  **at Big-Rip** while **for SFS** the scale factor **is finite** at  $\eta$ -time at SFS so that **the invariant string size is also finite**. The same is true for type III, IV and generalized SFS.

This means strings are **not destroyed** at weak singularities.

## Classification of singularities in FRW cosmology.

Type	Name	t sing.	a(t <sub>s</sub> )	ρ(t <sub>s</sub> )	p(t <sub>s</sub> )	ḡ(t <sub>s</sub> ) etc.	w(t <sub>s</sub> )	T	K
0	Big-Bang (BB)	0	0	∞	∞	∞	finite	strong	strong
I	Big-Rip (BR)	t <sub>s</sub>	∞	∞	∞	∞	finite	strong	strong
I <sub>l</sub>	Little-Rip (LR)	∞	∞	∞	∞	∞	finite	strong	strong
I <sub>p</sub>	Pseudo-Rip (PR)	∞	∞	finite	finite	finite	finite	weak	weak
II	Sudden Future (SFS)	t <sub>s</sub>	a <sub>s</sub>	ρ <sub>s</sub>	∞	∞	finite	weak	weak
II <sub>g</sub>	Gen. Sudden Future (GSFS)	t <sub>s</sub>	a <sub>s</sub>	ρ <sub>s</sub>	p <sub>s</sub>	∞	finite	weak	weak
III	Finite Scale Factor (FSF)	t <sub>s</sub>	a <sub>s</sub>	∞	∞	∞	finite	weak	strong
IV	Big-Separation (BS)	t <sub>s</sub>	a <sub>s</sub>	0	0	∞	∞	weak	weak
V	w-singularity (w)	t <sub>s</sub>	a <sub>s</sub>	0	0	0	∞	weak	weak

## Exotic singularities can mimic dark energy.

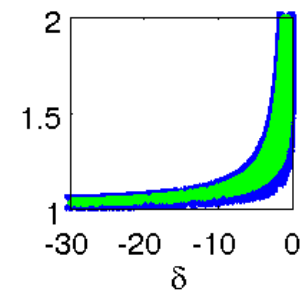
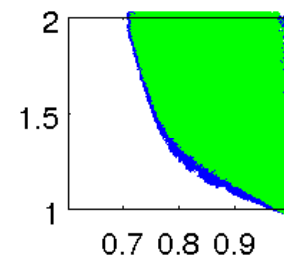
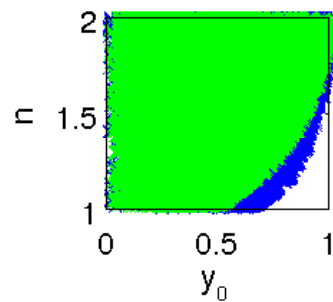
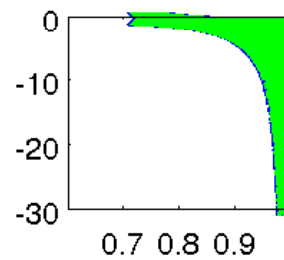
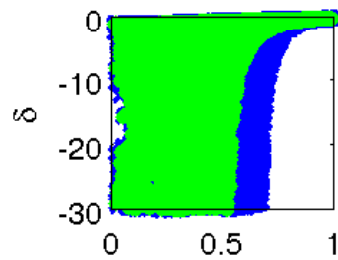
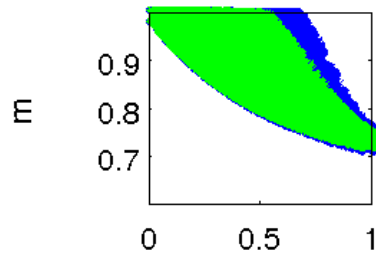


SFS - supernovae only (MPD et al. 2007): distance modulus  $\mu_L = m - M$  for the CC model ( $H_0 = 72 \text{ km s}^{-1} \text{ Mpc}^{-1}$ ,  $\Omega_{m0} = 0.26$ ,  $\Omega_{\Lambda 0} = 0.74$ ) (dashed curve) and SFS model ( $m = 2/3 = 0.6666$ ,  $n = 1.9999$ ,  $\delta = -0.471$ ,  $y_0 = 0.99936$ ) (solid curve). Open circles are for the ‘Gold’ data and filled circles are for SNLS data.

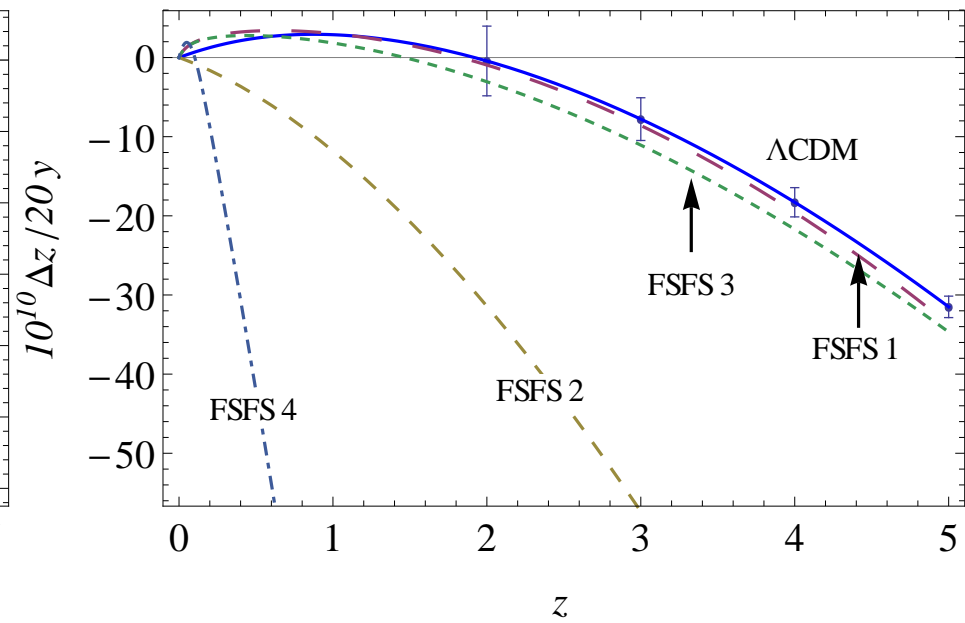
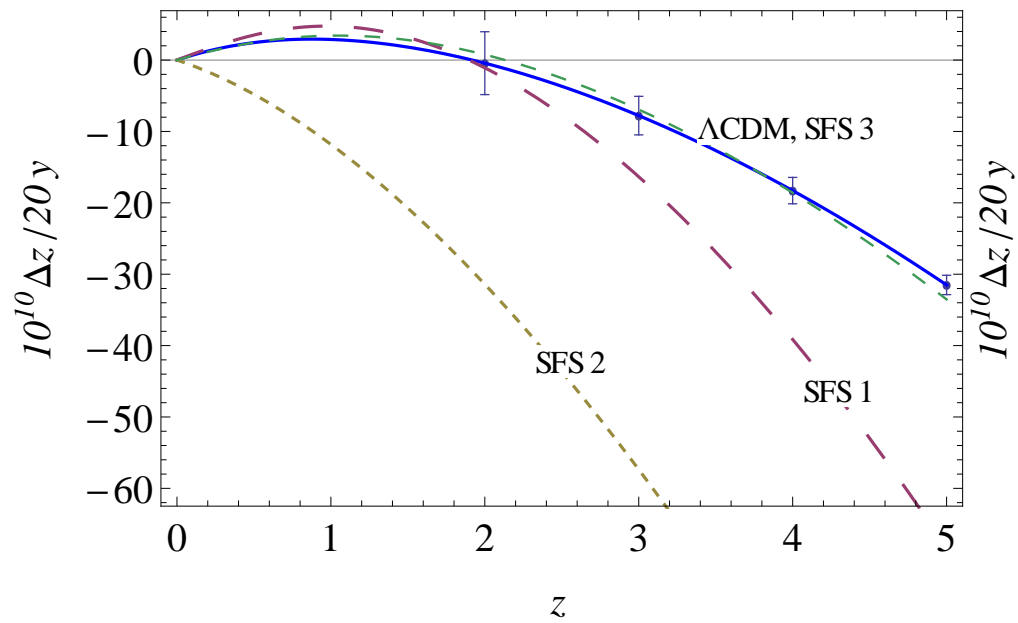
## SFS, FSFS: supernovae, CMB, BAO (Denkiewicz et al. 2012)

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Fits if  $m \approx 0.72$ ,  $w = -0.82$



## Another test: redshift drift (Denkiewicz, MPD, Martins, Vielzeuf, 2014).





## redshift drift - parameters

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	$m$	$\delta$	$n$	$t_0/t_s$
SFS 1	2/3	-0.43	1.9999	0.99
SFS 2	2/3	0.0	1.9999	0.99
SFS 3	0.749	-0.45	1.99	0.77
FSFS 1	0.56	0.42	0.8	0.96
FSFS 2	2/3	0.0	0.7	0.79
FSFS 3	2/3	0.24	0.7	0.96
FSFS 4	1.15	7.5	0.81	0.51

$H_0 = 67.3 \text{ km/s/Mpc}$  and  $\Omega_{m0} = 0.315$  (cf. Planck data 2013).

### 3. The universe through a singularity - averaging surprises.

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A.K. Raychaudhuri (PRL 80, 654 (1998)) proposed that one may average physical and kinematical scalars over the whole open spacetime provided they vanish rapidly at spatial and temporal infinity as follows

$$\langle \chi \rangle = \lim_{x^a \rightarrow \infty} \frac{\int \int \int \int_{-x^a}^{x^a} \chi \sqrt{-g} d^4 x}{\int \int \int \int_{-x^a}^{x^a} \sqrt{-g} d^4 x} \quad (27)$$

By an open model it is meant that the ratio of the 3-volume hypersurfaces to a 4-volume of spacetime vanishes, i.e.,

$$\frac{\int \int \int \sqrt{|^3 g|} d^3 x}{\int \int \int \int \sqrt{-g} d^4 x} = 0. \quad (28)$$

His idea was to tight the vanishing of the average  $\langle \chi \rangle$  with the singularity avoidance in cosmology.

## Spacetime averaging - density and pressure.

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For the pressure, the energy density, and the average acceleration we have

$$\langle p \rangle = - \lim_{\substack{t_0 \rightarrow 0 \\ t_1 \rightarrow \infty}} \frac{\int_{t_0}^{t_1} a^3 \left( 2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) dt}{\int_{t_0}^{t_1} a^3 dt} \quad (29)$$

and

$$\langle \rho \rangle = \lim_{\substack{t_0 \rightarrow 0 \\ t_1 \rightarrow \infty}} \frac{3 \int_{t_0}^{t_1} a^3 \left( \frac{\dot{a}^2}{a^2} \right) dt}{\int_{t_0}^{t_1} a^3 dt}. \quad (30)$$

$$\langle \dot{\theta} \rangle = \lim_{\substack{t_0 \rightarrow 0 \\ t_1 \rightarrow \infty}} \frac{3 \int_{t_0}^{t_1} a^3 \left( \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) dt}{\int_{t_0}^{t_1} a^3 dt}. \quad (31)$$

## SFS universe through an exotic singularity.

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One is able to construct a hybrid model which allows Big-Bang, Sudden Future Singularity and finally Big-Crunch given by:

$$a_L(t) = a_s \left[ \delta + \left(1 + \frac{t}{t_B}\right)^m (1 - \delta) - \delta \left(-\frac{t}{t_B}\right)^n \right] \quad (32)$$

with  $t_B < 0$  - the Big-Bang time, and  $t = 0$  and SFS time;

$$a_R(t) = a_s \left[ \delta + \left(1 - \frac{t}{t_C}\right)^m (1 - \delta) - \delta \left(\frac{t}{t_C}\right)^n \right] \quad (33)$$

with  $t_C > 0$  - the Big-Crunch time. In the high pressure regime  $t \rightarrow 0$  these are approximated by

$$a_L \approx a_s \left[ 1 + \frac{m}{t_B} (1 - \delta) t \right], \quad (34)$$

$$a_R \approx a_s \left[ 1 - \frac{m}{t_C} (1 - \delta) t \right]. \quad (35)$$

## Spacetime averaging - standard and phantom models

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$$\langle p \rangle_{stand} = \lim_{\substack{t_0 \rightarrow 0 \\ t_1 \rightarrow \infty}} -\frac{4}{\gamma} \left( \frac{1}{\gamma} - 1 \right) \frac{\int_{t_0}^{t_1} t^{2(\frac{1}{\gamma}-1)} dt}{\int_{t_0}^{t_1} t^{\frac{2}{\gamma}} dt} \rightarrow 0,$$

$$\langle \rho \rangle_{stand} = \lim_{\substack{t_0 \rightarrow 0 \\ t_1 \rightarrow \infty}} -\frac{4}{3\gamma^2} \frac{\int_{t_0}^{t_1} t^{2(\frac{1}{\gamma}-1)} dt}{\int_{t_0}^{t_1} t^{\frac{2}{\gamma}} dt} \rightarrow 0$$

$$\langle p \rangle_{ph} = \lim_{\substack{t_0 \rightarrow 0 \\ t_1 \rightarrow \infty}} -\frac{4}{|\gamma|} \left( \frac{1}{|\gamma|} + 1 \right) \frac{\int_{t_0}^{t_1} t^{-2(\frac{1}{|\gamma|}+1)} dt}{\int_{t_0}^{t_1} t^{-\frac{2}{|\gamma|}} dt} \rightarrow \infty,$$

$$\langle \rho \rangle_{ph} = \lim_{\substack{t_0 \rightarrow 0 \\ t_1 \rightarrow \infty}} -\frac{4}{3\gamma^2} \frac{\int_{t_0}^{t_1} t^{-2(\frac{1}{|\gamma|}+1)} dt}{\int_{t_0}^{t_1} t^{-\frac{2}{|\gamma|}} dt} \rightarrow \infty$$

## Spacetime averaging - SFS and FSF models

---

$$\dot{a}_L(t) = a_s \left[ \frac{m}{t_B} \left(1 + \frac{t}{t_B}\right)^{m-1} (1 - \delta) + \delta \frac{n}{t_B^n} (-t)^{n-1} \right] \quad (36)$$

$$\dot{a}_R(t) = a_s \left[ -\frac{m}{t_C} \left(1 - \frac{t}{t_C}\right)^{m-1} (1 - \delta) + \delta \frac{n}{t_C^n} (t)^{n-1} \right] \quad (37)$$

$$\frac{\ddot{a}_L}{a_s} = \frac{m(m-1)(1-\delta)}{t_B^2} \left(1 + \frac{t}{t_B}\right)^{m-2} - \frac{\delta n(n-1)}{t_B^n} (-t)^{n-2} \quad (38)$$

$$\frac{\ddot{a}_R}{a_s} = \frac{m(1-m)(1-\delta)}{t_C^2} \left(1 - \frac{t}{t_C}\right)^{m-2} + \frac{\delta n(n-1)}{t_C^n} t^{n-2} \quad (39)$$

Only the last terms blow up to give infinite pressure for  $1 < n < 2$  at  $t = 0$  so that we neglect other terms in  $a$ ,  $\dot{a}$  and  $\ddot{a}$ .

## Spacetime averaging - SFS and FSF models

---

$$\begin{aligned}
 \langle \dot{\theta} \rangle_{SFS,L} &= \lim_{\substack{t_0 \rightarrow -t_B \\ t_1 \rightarrow 0}} -3n \frac{\int_{t_0}^{t_1} (-t)^{3n-2} dt}{\int_{t_0}^{t_1} (-t)^{3n} dt} & (40) \\
 &= \lim_{\substack{t_0 \rightarrow -t_B \\ t_1 \rightarrow 0}} -3n \frac{3n+1}{3n-1} \frac{(-t_1)^{3n-1} - (-t_0)^{3n-1}}{(-t_1)^{3n+1} - (-t_0)^{3n+1}} \rightarrow \frac{1}{t_B^2}
 \end{aligned}$$

$$\begin{aligned}
 \langle \dot{\theta} \rangle_{SFS,R} &= \lim_{\substack{t_0 \rightarrow 0 \\ t_1 \rightarrow t_C}} 3n \frac{\int_{t_0}^{t_1} t^{3n-2} dt}{\int_{t_0}^{t_1} t^{3n} dt} & (41) \\
 &= \lim_{\substack{t_0 \rightarrow 0 \\ t_1 \rightarrow t_C}} 3n \frac{3n+1}{3n-1} \frac{t_1^{3n-1} - t_0^{3n-1}}{t_1^{3n+1} - t_0^{3n+1}} \rightarrow \frac{1}{t_C^2}
 \end{aligned}$$

These averages are finite for SFS, but they may blow up for FSF if  $0 < n < 1/3!$

## Subtle differences between singularities.

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- BB, BC singularities - all the energy conditions fulfilled, averages vanish (despite original claim of Raychaudhuri)
- BR singularity - no EC fulfilled, averages blow up
- SFS - only dominant energy violated, averages finite
- It seems that BR is stronger singularity than BB, BC on the ground of averaging.
- SFS is weaker, but FSF does not seem so.

This seems to be a new kind of a measure for the strength of singularities.



## 4. Varying constants theories.

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- It has been shown that **quantum effects** (e.g. Houndjo et al.: 1203.6084) may **change the strength** of exotic singularities (e.g. SFS to FSF).
- On the other hand, varying constants cosmologies have been applied to **solve standard cosmology problems** such as the horizon and flatness problem (e.g. Albrecht, Magueijo 1999).
- **Our idea** is to apply them to **solve the singularity problem** in cosmology.
- We can also ask if varying constants theories **can soften/strengthen** the standard and exotic singularities?

## varying constants theories

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First fully quantitative framework: **Brans-Dicke** scalar-tensor gravity (1961)

The gravitational constant  $G$  is associated with an average gravitational potential (scalar field)  $\phi$  surrounding a given particle:

$\langle \phi \rangle = GM/(c/H_0) \propto 1/G = 1.35 \times 10^{28} g/cm$ . The **scalar field gives the strength of gravity**

$$G = \frac{1}{16\pi\Phi} \quad (42)$$

With the action

$$S = \int d^4x \sqrt{-g} \left( \Phi R - \frac{\omega}{\Phi} \partial_\mu \Phi \partial^\mu \Phi + \Lambda + L_m \right) \quad (43)$$

it relates to low-energy-effective **superstring** theory for  $\omega = -1$

String coupling constant (running)  $g_s = \exp(\phi/2)$  changes in time with  $\phi$  - the **dilaton** and  $\Phi = \exp(-\phi)$ .

## varying constants theories

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Varying speed of light theories (VSL): Albrecht & Magueijo model (AM model) (1999)(Barrow 1999; Magueijo 2003):

$$c^4 = \psi(x^\mu) \quad (44)$$

and so the action is

$$S = \int d^4x \sqrt{-g} \left[ \frac{\psi(R + 2\Lambda)}{16\pi G} + L_m + L_\psi \right] \quad (45)$$

AM model **breaks Lorentz invariance** (relativity principle and light principle) - preferred frame (cosmological or CMB) in which the field is minimally coupled to gravity.

Solves basic problems of standard cosmology: horizon problem and flatness problem.

Ansatz: Friedmann with  $\rho = \rho_0 a^{-3\gamma}$ ,  $c(t) = c_0 a^n$  - solution if  $n \leq (1/2)(2 - 3\gamma)$ .

## varying constants theories

---

**Magueijo covariant** (conformally) and **locally invariant** model (2000, 2001):

$$\psi = \ln \left( \frac{c}{c_0} \right) \quad \text{or} \quad c = c_0 e^\psi, \quad (46)$$

with the action

$$S = \int d^4x \sqrt{-g} \left[ \frac{c_0^4 e^{\alpha\psi} (R + 2\Lambda + L_\psi)}{16\pi G} + e^{\beta\psi} L_m \right], \quad (47)$$

with

$$L_\psi = \kappa(\psi) \nabla_\mu \psi \nabla^\mu \psi. \quad (48)$$

Further assumption:  $\alpha - \beta = 4$ .

Interesting subcases:

$\alpha = 4; \beta = 0$  - Brans-Dicke with  $\phi_{BD} = e^{4\psi} / G$  and  $\kappa(\psi) = 16\omega_{BD}(\phi_{BD})$ .

$\alpha = 0; \beta = -4$  - minimal VSL theory.

## varying constants theories

---

Varying fine structure constant  $\alpha$  (or charge  $e = e_0\epsilon(x^\mu)$ ) theories (Webb et al. 1999, Sandvik 2002)

$$S = \int d^4x \sqrt{-g} \left( \psi R - \frac{\omega}{2} \partial_\mu \psi \partial^\mu \psi - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} e^{-2\psi} + L_m \right) \quad (49)$$

with  $\psi = \ln \epsilon$  and  $f_{\mu\nu} = \epsilon F_{\mu\nu}$ .

Assume linear expansion  $e^\psi = 1 - 8\pi G \zeta (\psi - \psi_0) = 1 - \Delta\alpha/\alpha$  with the constraint on the local equivalence principle violation  $|\zeta| \leq 10^{-3}$ . [The relation to dark energy is:](#)

$$\gamma = w + 1 = \frac{(8\pi G \frac{d\psi}{d \ln a})^2}{\Omega_\psi}. \quad (50)$$

While mimicking the dark energy this can be tested by spectrograph CODEX (COsmic Dynamics EXplorer) attached to the planned E-ELT (European Extremely Large Telescope) which measures the redshift drift (or Sandage-Loeb effect) for  $2 < z < 5$  (Vielzeuf and Martins 2012).

## Some observational constraints:

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- $|(dG/dt)/G| < 9 \cdot 10^{-13}/year$  - from primordial nucleosynthesis (Accetta et al. 1990);
- $|(dG/dt)/G| < 1.6 \cdot 10^{-12}/year$  - from helioseismology (Guenther et al. 1998);
- $|(dG/dt)/G| < (4 \pm 9) \cdot 10^{-13}/year$  - from lunar laser ranging (LLR) (Williams et al. 1996);
- $\Delta\alpha/\alpha = (3.85 \pm 5.65) \cdot 10^{-8}$  - from Oklo phenomenon (Shlyakhter 1976, Petrov et al. 2006);
- $\Delta\alpha/\alpha = (-8 \pm 16) \cdot 10^{-7}$  - from meteorite dating (long-lived beta decays) (Olive et al. 2003);
- $\Delta\alpha/\alpha = (+1.3 \pm 2.4) \cdot 10^{-6}$  - from quasar absorption spectra (Bonifacio et al. 2014);
- $\Delta\beta/\beta = (5.7 \pm 3.8) \cdot 10^{-5}$  ( $\beta = m_e/m_p$ ) - from quasar absorption spectra (Ivanchik et al. 2005).

## 5. Varying constants versus cosmic singularities.

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We consider the Friedmann universes in **varying speed of light (VSL)** theories and **varying gravitational constant G** theories as follows ( $\rho$  - mass density;  $\varepsilon = \rho c^2(t)$  - energy density in  $Jm^{-3} = Nm^{-2} = kgm^{-1}s^{-2}$ )

$$\rho(t) = \frac{3}{8\pi G(t)} \left( \frac{\dot{a}^2}{a^2} + \frac{kc^2(t)}{a^2} \right), \quad (51)$$

$$p(t) = -\frac{c^2(t)}{8\pi G(t)} \left( 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{kc^2(t)}{a^2} \right), \quad (52)$$

and the energy-momentum “conservation law” is

$$\dot{\rho}(t) + 3\frac{\dot{a}}{a} \left( \rho(t) + \frac{p(t)}{c^2(t)} \right) = -\rho(t) \frac{\dot{G}(t)}{G(t)} + 3\frac{kc(t)\dot{c}(t)}{4\pi Ga^2}. \quad (53)$$

## General form of the scale factor.

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We use a general form of the scale factor (MPD, K. Marosek, JCAP 02 (2013), 012), which **admits big-bang, big-rip, sudden future, finite scale factor and  $w$ -singularities** and reads as

$$a(t) = a_s \left( \frac{t}{t_s} \right)^m \exp \left( 1 - \frac{t}{t_s} \right)^n, \quad (54)$$

with the constants  $t_s, a_s, m, n$ . For  $k = 0$  we have

$$\rho(t) = \frac{3}{8\pi G(t)} \left[ \frac{m}{t} - \frac{n}{t_s} \left( 1 - \frac{t}{t_s} \right)^{n-1} \right]^2, \quad (55)$$

$$p(t) = -\frac{c^2(t)}{8\pi G(t)} \left[ \frac{m(3m-2)}{t^2} - 6\frac{mn}{tt_s} \left( 1 - \frac{t}{t_s} \right)^{n-1} \right. \\ \left. + 3\frac{n^2}{t_s^2} \left( 1 - \frac{t}{t_s} \right)^{2(n-1)} + 2\frac{n(n-1)}{t_s^2} \left( 1 - \frac{t}{t_s} \right)^{n-2} \right]. \quad (56)$$



## The scale factor - parametrization.

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For  $m < 0$  we have **a big-rip singularity** -  $a \rightarrow \infty$ ,  $\rho \rightarrow \infty$ ,  $p \rightarrow \infty$  at  $t = 0$ ;

For  $1 < n < 2$  we have **a sudden future singularity** (SFS) which appears at  $t = t_s$  ( $a = a_s$ ,  $\rho = \text{const.}$ ,  $p \rightarrow \infty$ );

For  $0 < n < 1$  we have **a (stronger) finite scale factor singularity** (FSF) at  $t = t_s$  ( $a = a_s$ ,  $\rho \rightarrow \infty$ ,  $p \rightarrow \infty$ ).

In fact, for  $1 < n < 2$  only the last term in the pressure of the type  $(1 - t/t_s)^{n-2}$  blows-up, while for  $0 < n < 1$  two more terms  $(1 - t/t_s)^{n-1}$  and  $(1 - t/t_s)^{2(n-1)}$  do.

# Regularizing singularities by varying constants

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One bears in mind the scale factor (54), the energy density (55) and pressure (56)

## Regularizing a Big-Bang singularity by varying $G$ :

If

$$G(t) \propto \frac{1}{t^2} \quad (57)$$

which is a faster decrease than in Dirac's LNH  $G \propto 1/t$ , but influences less the temperature of the Earth constraint (Teller 1948).

Both divergence in  $\rho$  and  $p$  are removed, though **at the expense of having the "singularity" of strong gravitational coupling  $G \rightarrow \infty$  at  $t \rightarrow 0$ .**

In the Dirac's case, only the  $\rho$  singularity can be removed.

## regularizing singularities by varying constants: SFS

---

### Regularizing an SFS singularity by varying $c$ :

If

$$c(t) = c_0 \left(1 - \frac{t}{t_s}\right)^{\frac{p}{2}}, \quad (58)$$

then

$$p(t) = -\frac{c_0^2}{8\pi G} \left[ \frac{m(3m-2)}{t^2} \left(1 - \frac{t}{t_s}\right)^p - 6\frac{mn}{tt_s} \left(1 - \frac{t}{t_s}\right)^{p+n-1} + 3\frac{n^2}{t_s^2} \left(1 - \frac{t}{t_s}\right)^{p+2n-2} + 2\frac{n(n-1)}{t_s^2} \left(1 - \frac{t}{t_s}\right)^{p+n-2} \right].$$

and the singularity of pressure is **regularized provided  $p > 2 - n$ , ( $1 < n < 2$ )**.

## regularizing singularities by varying constants: SFS.

---

Physical consequence: **light eventually stops** at the singularity. Same happens in loop quantum cosmology (LQC) where it is called the **anti-newtonian limit**  $c = c_0 \sqrt{1 - \rho/\rho_c} \rightarrow 0$  for  $\rho \rightarrow \rho_c$  with  $\rho_c$  being the critical density (Calettau et al. 2012). The **low-energy limit**  $\rho \ll \rho_0$  gives the standard limit  $c \rightarrow c_0$ .

It also appears naturally in **Magueijo model** ((Magueijo, PRD 63, 043502 (2001))) in which black holes are not reachable since the **light stops at the horizon** (despite they possess Schwarzschild singularity).

Strangely, both options  $c = 0$  and  $c = \infty$  are possible in Magueijo model.

## regularizing singularities by varying constants: $w$ -sing.

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In the limit  $m \rightarrow 0$  we have an exotic singularity scale factor given by  $a(t) = a_s \exp(1 - t/t_s)$  and so from (55) and (56) we have

$$\rho_{ex}(t) = \frac{3}{8\pi G(t)} \frac{n^2}{t_s^2} \left(1 - \frac{t}{t_s}\right)^{2(n-1)}, \quad (59)$$

$$p_{ex}(t) = -\frac{c^2(t)}{8\pi G(t)} \left[ 3 \frac{n^2}{t_s^2} \left(1 - \frac{t}{t_s}\right)^{2(n-1)} + 2 \frac{n(n-1)}{t_s^2} \left(1 - \frac{t}{t_s}\right)^{n-2} \right] \quad (60)$$

so that

$$w_{ex}(t) = \frac{p_{ex}(t)}{\varepsilon_{ex}(t)} = - \left[ 1 + \frac{2}{3} \frac{n-1}{n} \frac{1}{\left(1 - \frac{t}{t_s}\right)^n} \right] = - \left[ \frac{1}{3} - \frac{2}{3} q_{ex}(t) \right], \quad (61)$$

which is a  $w$ -singularity for  $n > 2$  ( $p = \rho = 0$ ,  $w_{ex} \rightarrow \infty$ ). Its regularization by varying  $c(t)$  is impossible since there is no  $c$ -dependence here.

# regularizing singularities by varying constants: SFS

## Regularizing an SFS singularity by varying $G$ :

If we assume that

$$G(t) = G_0 \left(1 - \frac{t}{t_s}\right)^{-r}, \quad (62)$$

( $r = \text{const.}$ ,  $G_0 = \text{const.}$ ) which changes (55) and (56) to

$$\begin{aligned} \rho(t) &= \frac{3}{8\pi G_0} \left[ \frac{m^2}{t^2} \left(1 - \frac{t}{t_s}\right)^r - \frac{2mn}{tt_s} \left(1 - \frac{t}{t_s}\right)^{r+n-1} \right. \\ &\quad \left. + \frac{n^2}{t_s^2} \left(1 - \frac{t}{t_s}\right)^{r+2n-2} \right], \end{aligned} \quad (63)$$

$$\begin{aligned} p(t) &= -\frac{c^2}{8\pi G_0} \left[ \frac{m(3m-2)}{t^2} \left(1 - \frac{t}{t_s}\right)^r - 6\frac{mn}{tt_s} \left(1 - \frac{t}{t_s}\right)^{r+n-1} + 3\frac{n^2}{t_s^2} \left(1 - \frac{t}{t_s}\right)^{r+2n-2} \right. \\ &\quad \left. + 2\frac{n(n-1)}{t_s^2} \left(1 - \frac{t}{t_s}\right)^{r+n-2} \right]. \end{aligned} \quad (64)$$

## regularizing singularities by varying constants: SFS

---

From (63) and (64) it follows that an SFS singularity ( $1 < n < 2$ ) is **regularized** by varying gravitational constant when

$$r > 2 - n , \quad (65)$$

and an FSF singularity ( $0 < 1 < n$ ) is **regularized** when

$$r > 1 - n . \quad (66)$$

On the other hand, assuming that we have an SFS singularity and that

$$-1 < r < 0 , \quad (67)$$

we get that varying  $G$  **may change an SFS singularity onto a stronger FSF singularity** when

$$0 < r + n < 1 . \quad (68)$$

## Regularizing singularities: (anti-)Chaplygin gas

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The equation of state of the (anti-)Chaplygin gas reads as

$$p(t) = \pm \frac{A}{\varepsilon(t)} = \pm \frac{A}{\rho(t)c^2(t)} \quad (A > 0) , \quad (69)$$

where the “-” sign is for Chaplygin gas while the “+” sign is for anti-Chaplygin gas case and the unit of  $A$  is the energy density(=pressure) square  $J^2 m^{-6}$ .

Inserting (69) into (53) gives

$$\dot{\rho}(t) + 3 \frac{\dot{a}}{a} \left( \frac{\rho^2 c^4(t) \mp A}{\rho(t) c^4(t)} \right) = -\rho(t) \frac{\dot{G}(t)}{G(t)} + 3 \frac{kc(t)\dot{c}(t)}{4\pi G(t)a^2} . \quad (70)$$

We assume both varying  $G = G(t)$  and  $c = c(t)$  though with zero curvature ( $k = 0$ ) as follows

$$\rho(t)c^2(t) = B = \text{const.} , \quad (71)$$



## regularizing singularities: (anti-)Chaplygin gas

---

The solution of (70) reads as

$$\varrho(t)a^{3\gamma}(t)G(t) = E = \text{const.} , \quad (72)$$

where we have defined

$$\gamma \equiv \frac{B^2 \mp A}{B^2} \quad (73)$$

Putting the standard big-bang scale factor  $a(t) = (t/t_s)^{2/3\gamma}$ , we finally have

$$\varrho(t) = \frac{Et_s^2}{t^2 G(t)} , \quad p(t) = \mp \frac{A}{B} = \text{const.} , \quad (74)$$

which give  $\varrho \rightarrow \infty$  and  $p(0) = 0$  provided  $G(0) = \text{const.} \neq 0$ . The singularity at  $t = 0$  in  $\varrho$  and  $p$  **can be regularized** by taking  $G(t) \propto 1/t^2$  at the expense of having a constant pressure (cosmological term) instead of zero pressure.

## Subtleties:

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- In order to regularize an SFS or an FSF singularity by varying  $c(t)$ , the **light should slow and eventually stop propagating** at a singularity. Similar effects were found in loop quantum cosmology (LQC) as well as in VSL for Schwarzschild horizon (Magueijo 2001) - speed of light is either zero or infinity at  $r = r_s$ . An observer cannot reach this surface even in his finite proper time.
- To regularize an SFS, FSF by varying gravitational constant  $G(t)$  - **the strength of gravity has to become infinite** at a singularity. On the one hand, it is quite reasonable because of the requirement to **overcome an infinite (anti-)tidal forces** at the singularity, but on the other hand, it makes another singularity - **a singularity of strong coupling** for a physical field such as  $G \propto 1/\Phi$ . Such problems were already dealt with in superstring and brane cosmology where both the curvature singularity and a strong coupling singularity appeared (choice of coupling, quantum corrections).

## 6. Conclusions

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- Currently one is able to **differentiate quite a number of cosmological singularities** with completely different properties - despite many of them are geodesically complete, they still lead to a **blow-up of physical quantities** such as scale factor, energy density, pressure, physical fields etc.
- Some of these singularities **may serve as dark energy**, especially if they are quite close in the near future. For example, **an SFS may even appear in 8.7 Myr** with no contradiction with bare supernovae data. It **can be fitted** to a combined SnIa, CMB, BAO, and redshift drift data for specific choice of the parameters. The same is true for FSF.
- Our proposal is to investigate **how the singularities are influenced by varying physical constants**. In particular, we may look for the answer if it is possible to **”regularize” (remove infinities) or change** these singularities and what are the physical consequences of this, because what we face is usually the **new ”singularity” in a physical fields** which act to remove/change a particular type of a singularity.